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EXTREMAL PSD FORMS WITH FEW TERMS

BRUCE REZNICK

1. Introduction

A psd form is a homogeneous polynomial p for which $p(x_1, \cdots, x_n) \geq 0$. Let $P_{n,2m}$ denote the convex cone of all psd forms in n variables with degree $2m$ and $\Sigma_{n,2m}$ denote the convex cone of all such forms which can be written as a sum of squares of forms. (It is clear that a sum of squares is psd.)

Hilbert [7] showed in 1888 that $\Sigma_{n,2m} = P_{n,2m}$ if and only if $(n, 2m)$ is $(n, 2)$, $(2, 2m)$ or $(3, 4)$ and that $\Sigma_{n,2m} \subset P_{n,2m}$ otherwise. He gave a method for constructing psd forms which are not a sum of squares, but did not carry it out. In fact, no explicit form in $P_{n,2m} - \Sigma_{n,2m}$ was exhibited until 1967.

Motzkin [9] demonstrated that

$$M(x_1, x_2, x_3) = x_1^6 + x_2^4x_3^2 + x_2^2x_3^4 - 3x_1^2x_2^2x_3^2$$

is such a form; the simplicity of M contrasts with the complexity of Hilbert's construction. Robinson [11] simplified Hilbert's method and provided several more such forms. Very recently Choi and Lam [1], [2], [3] have looked at $P_{n,2m}$ as a cone and searched for extremal elements. They proved that M, a number of Robinson's forms, and

$$S(x_1, x_2, x_3) = x_1^4x_2^2 + x_2^4x_3^2 + x_1^2x_3^4 - 3x_1^2x_2^2x_3^2$$

are all extremal psd forms in this sense.

The simplicity of M and S motivate this paper, in which all extremal psd forms with four or fewer terms (which are not sums of squares) will be described.

2. Preliminaries

Identify a form in n variables of degree m with the N-tuple of its coefficients ordered in any predetermined manner, where $N(n, m) = \binom{n+m-1}{n-1}$, and pull back the ordinary topology on $\mathbb{R}^N$. Then $P_{n,2m}$ is a closed cone. Ellison [5] has shown that $\Sigma_{n,2m}$ is also a closed cone. If f is extremal in $P_{n,2m}$ as a cone and $f = g_1 + g_2$, g_i psd, then $g_i = \lambda_i f$; if f is extremal in $\Sigma_{n,2m}$, then f is a perfect square. Let $E_{n,2m}$ consists of the extremal forms in $P_{n,2m}$ which are not perfect squares. We shall include the condition "not a perfect square" in any further use of the word "extremal". If $h = x_1^{a_1} \cdots x_n^{a_n}$, $\Sigma a_i = k$, and f is in $E_{n,2m}$ then h^2f is in $E_{n,2m+2k}$: if $x_j^{2a_j}$ divides $g_1 + g_2$, g_i psd, then $x_j^{2a_j}$ divides each g_i.

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