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APPLICATIONS OF A COMMUTATION FORMULA

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Let A and B be bounded operators in a Hilbert space. Then the following formula holds (see e.g., Sakai [27])

$$\lambda(AB + \lambda)^{-1} + A(BA + \lambda)^{-1}B = 1 \tag{1}$$

in the sense that if $0 \neq -\lambda \in \rho(BA)$ then $-\lambda \in \rho(AB)$ and $\lambda^{-1}(1 - A(BA + \lambda)^{-1}B)$ gives $(AB + \lambda)^{-1}$, the resolvent of AB. Interchanging $A \leftrightarrow B$ we have immediately

$$\sigma(AB)\setminus\{0\} = \sigma(BA)\setminus\{0\}. \tag{2}$$

We will call (1), the *commutation formula* and we will refer to a result developed from (1) or (2) as being based on *commutation*. Our aim in this paper is to show that a wide spectrum of problems in mathematical physics are in fact based on commutation. It will be apparent in some of the older problems we discuss that not only the result but also the old method is based on (1) or (2).

The paper is organized as follows:

In Section 1 we give a proof of an extended version of the commutation formula which includes the case where A is a closed operator and $B = A^*$, its adjoint. In Section 2 we give a (mini-) application of (2) to the theory of representations for the canonical commutation relations. In Section 3 we give a commutation proof of two basic results (Segal's lemma and the Golden-Thompson rule) in statistical mechanics and field theory.

In Section 4 we apply commutation to ordinary differential operators. The application is based on the following observation: let $V = V(x)$ be a potential and suppose $b = b(x)$ solves $-b'' + Vb = 0$ i.e., $V = b^{-1}(b'')$. Then direct computation gives the operator equality

$$\left(b \frac{d}{dx} b^{-1}\right)^*\left(b \frac{d}{dx} b^{-1}\right) = -b^{-1}\frac{d}{dx} b^2 \frac{d}{dx} b^{-1} = -\frac{d^2}{dx^2} + V(x).$$

By commutation, $(b(d/dx)b^{-1})(b(d/dx)b^{-1})^* = -b(d/dx)b^{-2}(d/dx)b$ has the same spectrum as $(b(d/dx)b^{-1})^*(b(d/dx)b^{-1})$, with the possible exception of zero. But $-b(d/dx)b^{-2}(d/dx)b = -(d^2/dx^2) + \tilde{V}(x)$, where $\tilde{V}(x) = b(b^{-1})''$. Hence, given a potential $V(x)$, there exists another potential $\tilde{V}(x)$ for which $-d^2/dx^2 + V(x)$ and $-d^2/dx^2 + \tilde{V}(x)$ (with suitable boundary conditions) have the same spectrum away from zero. Furthermore $\tilde{V} = V - 2(d^2/dx^2)\log b$.

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