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ASYMPTOTICS OF EIGENVALUE CLUSTERS FOR THE LAPLACIAN PLUS A POTENTIAL

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0. Introduction

Let $\triangle$ be the Laplace-Beltrami operator on a compact riemannian manifold X, $V: X \to \mathbb{R}$ a smooth function which we consider as a multiplication operator. The sum $H = \triangle + V$ is called a (reduced) Schrödinger operator, and its eigenvalues represent the energy levels of a quantum mechanical system with kinetic energy given by the riemannian metric, and potential energy V.

If X is the unit n-sphere S^n, the eigenvalues of $\triangle$ are of the form $k(k + n - 1)$, with multiplicities growing as a polynomial of order $n - 1$ in k. These are the energy levels of a "free particle." If a force field with potential V is now applied, each of the multiple eigenvalues splits into a "cluster" of eigenvalues in the interval

$$[k(k + n - 1) + \min V, k(k + n - 1) + \max V].$$

(The Stark effect, in which V is an electrostatic potential, is a physical example of this phenomenon.)

The structure of the clusters has recently been studied by Guillemin in a series of papers [G1][G2][G3] intended to show how the potential function V might be determined by the eigenvalues of $\triangle + V$. In this paper, we extend some of Guillemin's analysis by showing that the distribution of eigenvalues in the k'th cluster approaches a limit as $k \to \infty$, and that the limiting distribution can be expressed in terms of the averages of V along closed geodesics.

Although we begin with differential operators, our constructions immediately require the use of pseudodifferential operators, so we begin in that context as well. In section 1 we show that, modulo a small error, we can replace the potential V by a pseudodifferential operator which commutes with $\triangle$. Section 2, a study of eigenvalues, is a bridge to the principal Section 3, in which we analyze the joint spectrum of two commuting operators. Theorem 3.4 in that section is the main theorem in this paper.* In Section 4, we present some examples, including an application to the spectrum of manifolds all of whose geodesics are closed.

The aforementioned papers of Guillemin, as well as the numerical calculations of Chachere [C] (discussed in section 4) were the main impetus behind the

*M. Kac and T. Spencer, as well as H. Widom, have independently proven slightly weaker versions of Theorem 3.4, using completely different methods.

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