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ON FREE NILMANIFOLDS AND THEIR ASSOCIATED NON COMPACT SOLVMANIFOLDS

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Introduction. In their paper [2], Auslander and Tolimieri proved that any non compact solvmanifold (homogeneous space of a solvable Lie group) is diffeomorphic to the total space of a vector bundle over a compact solvmanifold. This result was refined in [3] by determining exactly which vector bundles over a compact solvmanifold have total spaces which are solvmanifolds. Some progress was also made on the classification problem. In particular, the classification of all solvmanifolds with abelian fundamental group was achieved. As it turned out, the number of solvmanifolds (without euclidean factors) with fundamental group free abelian of rank n is $\frac{1}{2}(3n + 2)$ if n is even and $\frac{1}{2}(3n + 1)$ if n is odd.

In this paper, we continue the study of non compact solvmanifolds determining up to diffeomorphism all solvmanifolds with free nilpotent fundamental group. (See section 1 for definitions.) In contrast with the abelian case, there are only three manifolds (without euclidean factors) with a given free nilpotent fundamental group. In addition, two of these become diffeomorphic after taking products with a euclidean space of the appropriate dimension. (See the corollary to Lemma 2.3.)

As in the abelian case, the classification referred to above depends on the equivalence problem for certain bundles over free nilmanifolds (defined in section 2). The basic result here is Lemma 2.3 which states that any bundle over a free nilmanifold which is the sum of at least three line bundles is equivalent to the sum of a line bundle and a trivial bundle.

The organization of the paper is as follows. Section 1 contains results about free nilpotent groups and Lie algebras and the relationship between them. In section 2, we define free nilmanifolds and solvmanifolds and state, with some proofs, the principal results of the paper. The last two sections contain proofs of lemmas needed in section 2.

1. Free Nilpotent Groups and Lie algebras. In this section, we define free nilpotent groups and Lie algebras, develop some of their elementary properties and determine the relationship between them. With the possible exception of Theorems 1.6 and 1.8, the material in this section is known.

Let Γ be a nilpotent group of class k (all $(k + 1)$-fold commutators are trivial)

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