K-REFLEXIVITY IN FINITE DIMENSIONAL SPACES

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Let $\mathcal{H}$ be an n-dimensional Hilbert space, $n \geq 4$. A subalgebra $\mathfrak{A}$ of $\mathcal{L}(\mathcal{H})$ is defined to be *k-reflexive* if $\mathfrak{A}^{(k)}$, a k-fold copy of $\mathfrak{A}$, is reflexive. The main theorems of the paper state that (1) every subalgebra of $\mathcal{L}(\mathcal{H})$ is $(n - 1)$-reflexive and (2) every commutative subalgebra of $\mathcal{L}(\mathcal{H})$ is $(n/2)$-reflexive. Examples are given to show these results are "best-possible".

Several alternate characterizations of k-reflexivity provide the main technique of the paper. As a by-product, we are able to exhibit a commutative algebra $\mathfrak{A}$ such that $\mathfrak{A} \neq \mathfrak{A}'' \cap \text{Alg Lat } \mathfrak{A}$; this answers a question of P. Rosenthal.

1. Introduction. All Hilbert spaces discussed in this paper will be complex and finite dimensional. We denote by $\mathcal{L}(\mathcal{H})$ the collection of all (linear) operators on the Hilbert space $\mathcal{H}$. Let $\mathfrak{A}$ be an identity-containing subalgebra of $\mathcal{L}(\mathcal{H})$. Then *Lat* $\mathfrak{A}$ denotes the collection of subspaces of $\mathcal{H}$ invariant under each operator in $\mathfrak{A}$ and *Alg Lat* $\mathfrak{A}$ denotes the collection of operators in $\mathcal{L}(\mathcal{H})$ leaving each subspace in Lat $\mathfrak{A}$ invariant. Finally, for $A \in \mathcal{L}(\mathcal{H})$ we write $A^{(k)}$ for the direct sum of k copies of A and $\mathfrak{A}^{(k)}$ for $\{A^{(k)} \mid A \in \mathfrak{A}\}$.

A subalgebra $\mathfrak{A}$ of $\mathcal{L}(\mathcal{H})$ is said to be *reflexive* if it is determined by its invariant subspaces, i.e., $\mathfrak{A} = \text{Alg Lat } \mathfrak{A}$. In [3] Deddens and Fillmore gave necessary and sufficient conditions for $\mathfrak{A}_A$, the algebra generated by A (and the identity), to be reflexive. The more general problem of determining all reflexive subalgebras of $\mathcal{L}(\mathcal{H})$ has so far defied solution.

The purpose of this paper is to measure how nonreflexive an algebra can be. More precisely, we call an algebra $\mathfrak{A}$ *k-reflexive* if $\mathfrak{A}^{(k)}$ is reflexive. Let $\mathcal{H}$ be n-dimensional, $n \geq 3$. Our main results follow.

(1) There exists a subalgebra of $\mathcal{L}(\mathcal{H})$ which is not $(n - 2)$-reflexive.

(2) Every subalgebra of $\mathcal{L}(\mathcal{H})$ is $(n - 1)$-reflexive.

These results are obtained in Section 3; for the proofs, we rely on several alternate characterizations of k-reflexivity obtained in Section 2.

The final three sections of the paper are devoted to extensions of the above results. In Section 4 we present analogues of (1) and (2) for commutative subalgebras of $\mathcal{L}(\mathcal{H})$; as a corollary, we see that not every commutative algebra $\mathfrak{A}$ satisfies $\mathfrak{A} = \mathfrak{A}'' \cap \text{Alg Lat } \mathfrak{A}$, thus answering a question of [6]. A generalization of (2) is proved in Section 5, and the paper closes with a discussion of vector spaces over fields other than **C**.

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