FOURIER TRANSFORMS AND CHAINS OF INNER FUNCTIONS

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1. Introduction. Let $\{\mathfrak{M}_t : 0 \leq t \leq \infty\}$ be a family of subspaces of a separable Hilbert space H. We will say that this family is a *continuous chain* if the following hold.

(i) $\mathfrak{M}_s \subset \mathfrak{M}_t$ if $s \leq t$.

(ii) $\mathfrak{M}_0 = \{0\}$ and $\bigcup_{t \geq 0} \mathfrak{M}_t$ is dense in H.

(iii) For each $s \geq 0$, $\bigcup_{t<s} \mathfrak{M}_t$ is dense in $\mathfrak{M}_s$ and $\bigcap_{t>s} \mathfrak{M}_t = \mathfrak{M}_s$.

For convenience we will further assume that

(iv) $\mathfrak{M}_s \neq H$ for $s < \infty$.

Two continuous chains $\{\mathfrak{M}_t\}_{t \geq 0}$ in H and $\{\mathfrak{N}_t\}_{t \geq 0}$ in K are *unitarily equivalent* if there exists a unitary operator $U : H \to K$ with $U\mathfrak{M}_t = \mathfrak{N}_t$, $t \geq 0$.

It follows from spectral multiplicity theory that, given a continuous chain $\{\mathfrak{M}_t\}_{t \geq 0}$ in H, there exists a direct integral of Hilbert spaces

$$\mathfrak{D} = \int_0^\infty \oplus H_t \, dm(t)$$

(see [4]) and a unitary operator $\mathfrak{F} : H \to \mathfrak{D}$ such that

$$\mathfrak{F}\mathfrak{M}_t = \chi_{[0,t]}\mathfrak{D} = \{\chi_{[0,t]}f : f \in \mathfrak{D}\}, \qquad t \geq 0.$$

Here $\chi_{[0,t]}$ is the characteristic function of $[0, t]$. The equivalence class $[m]$ of the scalar spectral measure m under the equivalence relation of mutual absolute continuity together with the m-a.e. determined multiplicity function $n(t) = \dim H_t$ form a complete set of unitary invariants for $\{\mathfrak{M}_t\}_{t \geq 0}$.

In this note we shall describe m, $n(t)$ and $\mathfrak{F}$ when $\{\mathfrak{M}_t\}_{t \geq 0}$ is any continuous chain of star-invariant subspaces of the Hardy space H^2. Necessarily $\mathfrak{M}_t = (\phi_t H^2)^\perp$, $t \geq 0$, where ϕ_t is a singular inner function.

We will refer to the operator $\mathfrak{F}$ as a *Fourier transform* for reasons made clear by an example. Section 2 contains the construction of $\mathfrak{F}$ and our main results. With certain special choices of $\{\phi_t\}_{t \geq 0}$ we obtain some earlier results of Ahern and Clark [1], Berger and Coburn [2], and the present author [6]. These are presented as examples in Section 3. In Section 4 the results of Section 2 are applied to characterize those chains $\{\phi_t\}_{t \geq 0}$ of singular inner functions whose linear span is dense in H^2. We show in Section 5 that the invariant subspace lattice of the simple unilateral shift contains a copy of every continuous chain of subspaces.

This paper arose from an attempt to understand the relationship between

Received August 31, 1972. This research was supported in part by NSF Grant GP 31483X.