SOME CONTINUED FRACTION FORMULAS

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1. Put

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n), \qquad (q)_0 = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k(q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}.$$

Hirschhorn [7] has proved that

$$1 + \frac{aq}{1+}\,\frac{aq^2}{1+}\cdots\frac{aq^n}{1} = \frac{P_n(a,q)}{Q_n(a,q)}, \tag{1.1}$$

where

$$P_n(a,q) = \sum_{2r\le n+1} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} a^r q^{r^2} \tag{1.2}$$

and

$$Q_n(a,q) = \sum_{2r\le n} \begin{bmatrix} n-r \\ r \end{bmatrix} a^r q^{r(r+1)}. \tag{1.3}$$

If $n \to \infty$ and $|q| < 1$, it is evident that (1.1) becomes the well-known result [6; Chapter 19]

$$1 + \frac{aq}{1+}\,\frac{aq^2}{1+}\cdots = \sum_{n=0}^{\infty} \frac{a^r q^{r^2}}{(q)_r} \Big/ \sum_{r=0}^{\infty} \frac{a^r q^{r(r+1)}}{(q)_r}. \tag{1.4}$$

It is also clear from (1.2) and (1.3) that

$$Q_n(a,q) = P_{n-1}(aq,q). \tag{1.5}$$

Moreover

$$\begin{cases} P_r(a,q) = P_{r-1}(a,q) + aq^r P_{r-2}(a,q) \\ Q_r(a,q) = Q_{r-1}(a,q) + aq^r P_{r-2}(a,q) \end{cases} \tag{1.6}$$

for $r \geq 2$.

2. In view of (1.6) it may be of interest to consider finite continued fractions suggested by other sets of polynomials satisfying recurrences of the second order. A particularly simple set that has received a good deal of attention is defined by

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