PRODUCTS OF SHIFTS

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A *shift* on a Hilbert space H is an isometry S on H such that for some subspace K of H the subspaces K, SK, S^2K, $\cdots$ are pairwise orthogonal and span H. (This definition describes *unilateral* shifts only; the other kind, bilateral shifts, will be referred to by their full name.) The *multiplicity* of S is the dimension of K; since K is the co-range of S, the multiplicity of a shift is the same as its co-rank.

Every isometry on H is either a unitary operator, or a shift, or a direct sum of two operators of those two kinds. The set of all isometries on H is a semigroup with somewhat mysterious properties; the purpose of this paper is to illuminate a small corner of the algebraic theory of that semigroup.

Which operators on H have the form US, where U is unitary and S is a shift? All that is obvious is that every operator of that form is an isometry. Is it a shift? Which operators on H have the form S_1S_2, where S_1 and S_2 are shifts? Once again it is obvious that every operator of that form is an isometry. Is it a shift?

The answers are not deep, but they are somewhat surprising, and the techniques shed at least a little light on the chicanery of shifts. The heart of the matter is in the separable case, and that is treated first; afterward the general case is reduced to the separable one by considerations of cardinal arithmetic.

THEOREM 1. *On a separable Hilbert space, every isometry is either a unitary operator, or a shift, or a product of two operators of those two kinds.*

THEOREM 2. *On a separable Hilbert space, every isometry of co-rank at least* 2 *is a product of two shifts.*

Proof of Theorem 1. It is to be proved that if U is unitary, with $1 \leq \text{size } U \leq \aleph_0$ (the *size* of an operator is the dimension of its domain), and if S is a shift, with $1 \leq \text{mult } S \leq \aleph_0$ ("mult" stands for multiplicity), then the direct sum $U \oplus S$ is a product of a unitary operator and a shift.

It is sufficient to treat the case in which mult $S = 1$. Indeed, if mult $S > 1$, then express S as a direct sum of shifts of multiplicity 1, and apply the theorem to the direct sum of U and one of the direct summands of S. To obtain the result for $U \oplus S$, form the direct sum of the unitary factor that the theorem yields and an identity operator of size $\aleph_0$ for each unused summand of S, form the direct sum of the shift factor that the theorem yields and the unused

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