FACTORIZATIONS OF BOUNDED HOLOMORPHIC FUNCTIONS

P. R. AHERN AND WALTER RUDIN

1. Introduction. For a bounded holomorphic function f in the open unit disc we have a unique factorization $f = Bg$, where B is a Blaschke product and g is never 0. The Blaschke product B is a product of "irreducible" factors corresponding to the zeros of f, and g is uniquely determined by a certain measure on the unit circle. In this paper we consider bounded holomorphic functions in the polydisc U^n. In Section 2 we show that every bounded holomorphic function in U^n is uniquely determined by its zero set and a certain boundary measure. (We show that this holds for a larger class of functions.) In Section 3 we show that any bounded holomorphic function f in U^n can be factored $f = h \cdot \prod_{i=1}^{\infty} g_i$, where h is never 0 and each g_i is "irreducible" in a sense defined below. Unfortunately, this factorization need not be unique even if f is an inner function (see definitions below). Another difference between this and the one-variable case is that the irreducible factors g_i may have non-zero boundary measures associated with them. In the fourth section we construct examples of functions with non-unique factorizations.

We use the notation of [2]. $\mathbb{C}$ denotes the complex numbers, U the open unit disc in $\mathbb{C}$ and T the boundary of U. Let n be a positive integer. If μ is a real Borel measure on $T^n \subseteq C^n$, we denote its Poisson integral by $P[d\mu]$ and its Fourier transform by $\hat{\mu}$. We say $\mu \in RP(T^n)$ if $P[d\mu] \in RP(U^n)$, the space of real parts of functions holomorphic in U^n. This happens if and only if $\hat{\mu} \equiv 0$ outside of $Z_+^n \cup Z_-^n$, where $k = (k_1, \cdots, k_n) \in Z_+^n$ if $k_i \geq 0$ for all i and $Z_-^n = -Z_+^n$.

If f is holomorphic in U^n, we say $f \in H^\infty(U^n)$ if it is bounded in U^n, we say $f \in N(U^n)$ if $\log^+ |f|$ has an n-harmonic majorant in U^n, and we say $f \in A(U^n)$ if f is continuous on $\bar{U}^n$; f_r denotes the function whose value at z is $f(rz)$. If $f \in N(U^n)$, then $\log |f|$ has a *least* n-harmonic majorant denoted by $u[f]$. If $f \in N(U^n)$, then $\lim_{r\to 1} f(rz) = f^*(z)$ exists a.e. with respect to Haar measure m_n on T^n. If $g \in H^\infty(U^n)$ and $|g^*| = 1$ a.e., we say g is inner. If g is inner and $u[g] = 0$, we say g is good.

2. An observation about the Nevanlinna class.

2.1. Every $f \in N(U^n)$, $f \not\equiv 0$, determines two pieces of data; the first, obviously, is its zero set (including multiplicities), and the second is what we shall call its *boundary measure* β_f. This is a real Borel measure on T^n which may be defined

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