PERTURBATION OF NON-NORMAL ISOMETRIES

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1. Introduction. Let H be a separable Hilbert space. Denote by ker U the null space of an operator U and by null U the dimension of ker U. The following theorem will be proved.

THEOREM. *If U is a non-normal isometry on H, if S is a strict contraction, $||S|| < 1$, and if K is compact, then* ker $(U^* - S - K) \neq 0$; *if* null $U^* < \infty$, *then* null $(U^* - S - K) < \infty$.

In case S is a scalar (of modulus strictly less than 1) the result implies Stampfli's theorem proved in [4] and generalized in [1].

The strict inequality $||S|| < 1$ cannot be weakened to $||S|| \leq 1$; indeed if U is the unilateral shift and if $S = 1$ and $K = 0$, then ker $(U^* - S - K) =$ ker $(U^* - 1) = 0$.

2. Proof of the theorem. Observe that $U^* - S - K = U^*(1 - US - UK)$. Since $||US|| < 1$, it follows that $1 - US$ is invertible; since UK is compact, the Fredholm alternative implies that either ker $(1 - US - UK) \neq 0$ (In this case the first assertion of the theorem is obviously true.) or $1 - US - UK$ is invertible (In this case the first assertion of the theorem follows from the fact that ker $U^* \neq 0$.).

The second assertion is a consequence of the inequality null $(AB) \leq$ null $A +$ null B, with $A = U^*$ and $B = 1 - US - UK$. Indeed, by assumption null A is finite; since ker $B =$ ker $(1 - US)(1 - (1 - US)^{-1}UK) =$ ker $(1 - (1 - US)^{-1}UK)$ is the eigenpace of the eigenvalue 1 for the compact operator $(1 - US)^{-1}UK$, it follows that null B also is finite.

COROLLARY. *If every element in the null space of $U^* - S - K$ cannot be orthogonal to the null space of U^*, then* null $U^* =$ null $(U^* - S - K)$.

Proof. In this case the operator $1 - (1 - US)^{-1}UK$ is invertible. In fact since $(1 - US)^{-1}UK$ is compact, non-invertibility means that there exists at least one element f in H such that

$$(1) \qquad (1 - (1 - US)^{-1}UK)f = 0,$$

where f belongs to the null space of $U^* - S - K$. But from (1) it follows that $U(S + K)f = f$ and that $(f, f_0) = 0$ for every f_0 in the null space of U^*; this contradicts the assumption. Therefore null $(U^* - S - K) =$ null $(U^*(1 - US)(1 - (1 - US)^{-1}UK)) =$ null U^*.

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