EXTENDING SELF HOMEOMORPHISMS OF A WILD SPHERE DESCRIBED BY GILLMAN

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We define a 2-sphere S to be homogeneously embedded in E^3 (or S^3) if for each pair of points $p, q \in S$ there is a homeomorphism h of E^3 onto itself such that $h(S) = S$ and $h(p) = q$. The following question remains unanswered. Is a 2-sphere tame in E^3 if it is homogeneously embedded in E^3 [5; 28], [2], [4; 312]? Bing [1] described a 2-sphere S in S^3 which is wild at each of its points such that each arc in S is tame, and Gillman [6] modified Bing's example so that each component of $S^3 - S$ is topologically an open 3-cell. In the Proceedings of the University of Georgia Institute on the Topology of 3-manifolds in 1961, the question was raised whether either of the spheres described by Bing and Gillman offers a negative answer to the above question [5; 28].

Our purpose in this paper is to show that the 2-sphere described by Gillman [6] is not homogeneously embedded in E^3. We suspect that the one described by Bing [1] likewise is not homogeneously embedded in E^3, but the methods we use with Gillman's example do not apply to Bing's. It has previously been observed [4; 312] that neither of the spheres described by Bing and Gillman satisfies a stronger form of homogeneous embedding where each homeomorphism of the sphere onto itself can be extended to E^3.

The definitions we use will be similar to those in [3] and [7]. We let ϵ be a positive number, S a 2-sphere in E^3, and U a component of $E^3 - S$. We say that S can be *ϵ-spanned* in U on the simple closed curve J if there exists an ϵ-disk D in S, with $\text{Bd } D = J$, such that for each positive number δ there is an ϵ-disk D' in U with $\text{Bd } D'$ homotopically within δ of J. We say that S can be *locally spanned in U on tame simple closed curves* at the point $p \in S$ if for each $\epsilon > 0$, there exist ϵ-disks D and D' satisfying the above requirements such that $\text{Bd } D$ is tame and $p \in \text{Int } D$. If S has this property at each of its points, we say that S can be *locally spanned in U on tame simple closed curves.* Loveland [7] has shown that a 2-sphere S is tame in E^3 if it can be locally spanned on tame simple closed curves in each component of $E^3 - S$. We show that the wild sphere described by Gillman [6] is not homogeneously embedded in E^3 by showing that the contrary would imply that it can be locally spanned, in each of its complementary domains, on tame simple closed curves.

THEOREM 1. *If the 2-sphere S is homogeneously embedded in E^3 and for each $\epsilon > 0$ and each component U of $E^3 - S$ the sphere S can be ϵ-spanned in U on some tame simple closed curve, then S is tame.*

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