THE NUMBER OF $n \times n$ MATRICES OF RANK r AND TRACE α OVER A FINITE FIELD

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1. Introduction. Let $GF(q)$ denote a finite field of order $q = p^v$, p a prime. Let n be a positive integer, r an integer such that $0 \leq r \leq n$, and α an element of $GF(q)$. The purpose of this paper is to determine the number $N(n, q, r, \alpha)$ of $n \times n$ matrices of rank r and trace α over $GF(q)$. Since similar matrices have the same rank and trace, the first approach used by the author in an attempt to find $N(n, q, r, \alpha)$ was to consider canonical forms under similarity transformations. It appeared that in order to use this approach it would be necessary either to know all irreducible polynomials over a given finite field or to express the number $N(n, q, r, \alpha)$ in terms of the elements of a field $GF(q^m)$, an extension of $GF(q)$. Even if the latter approach had been feasible, it would have been necessary to express $N(n, q, r, \alpha)$ in terms of an expression containing summations extending over certain highly restricted partitions of the integer n.

In order to avoid the above difficulties, a difference equation in $N(n, q, r, \alpha)$, which appears in Section 3, was obtained. In Section 4 a solution to this difference equation is found.

2. Notation and preliminaries. Throughout this paper $A, B, \cdots$ will denote matrices over $GF(q)$. For a given matrix A, $\mathcal{RS}[A]$ will denote the row space of A and $\mathcal{CS}[A]$ will denote the column space of A. Let $g(s, t)$ denote the number of $s \times s$ matrices of rank t over $GF(q)$. Landsberg [1] has found this number to be

$$g(s, t) = q^{t(t-1)/2} \prod_{i=1}^{t} \frac{(q^{s-i+1} - 1)^2}{(q^i - 1)}. \tag{2.1}$$

Further, let $\mathcal{B}(n, q, r, \alpha)$ denote the set of all $n \times n$ matrices of rank r and trace α over $GF(q)$.

3. A difference equation in $N(n, q, r, \alpha)$. Let B be any element of $\mathcal{B}(n, q, r, \alpha)$, $1 \leq r \leq n$. Then clearly B may be expressed as

$$B = \begin{bmatrix} A & C \\ D & e \end{bmatrix}, \tag{3.1}$$

where A is an $(n - 1) \times (n - 1)$ matrix of rank r, $r - 1$, or $r - 2$, C and D^T are $(n - 1) \times 1$ vectors, and e is in $GF(q)$. Let $M_1(n, q, r, \alpha)$ denote the number

Received April 6, 1972. Revisions received June 1, 1972.