ABSOLUTE CONVERGENCE OF FOURIER SERIES ON CERTAIN GROUPS

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Let G be a compact, metrizable, 0-dimensional, abelian group. Then the dual group X of G is a countable, discrete, abelian, torsion group. In 1947 N. Ja. Vilenkin developed part of the Fourier theory on G. The main result of the present paper is a proof of a sufficient condition for the absolute convergence of a Fourier series on G. As a consequence we obtain the analogue on G of some well-known theorems for trigonometric Fourier series. Furthermore, we investigate the relationship between the class of functions on G with absolutely convergent Fourier series and classes of functions on G that satisfy a Lipschitz condition.

1. Let G and X be as above. Vilenkin [11] proved the existence of an increasing sequence $\{X_n\}$ of finite subgroups of X and of a sequence $\{\varphi_n\}$ of characters in X such that the following hold.

(i) $X_0 = \{\chi_0\}$, where $\chi_0(x) = 1$ for all $x \in G$.

(ii) For each $n \geq 1$, X_n/X_{n-1} is of prime order p_n.

(iii) $X = \bigcup_{n=0}^{\infty} X_n$.

(iv) $\varphi_n \in X_{n+1} \backslash X_n$ for all $n \geq 0$.

(v) $\varphi_n^{p_{n+1}} \in X_n$ for all $n \geq 0$.

Using these φ_n we enumerate the elements of X as follows. Let $m_0 = 1$ and let $m_n = \prod_{i=1}^{n} p_i$. If $k \geq 1$ and if $k = \sum_{i=0}^{s} a_i m_i$, with $0 \leq a_i < p_{i+1}$ if $0 \leq i \leq s$, then $\chi_k = \varphi_0^{a_0} \cdot \cdots \cdot \varphi_s^{a_s}$. Then $X_n = \{\chi_i : 0 \leq i < m_n\}$. Next, if G_n is the annihilator of X_n, that is,

$$G_n = \{x \in G : \chi_k(x) = 1 \quad \text{for all} \quad \chi_k \in X_n\},$$

then obviously $G = G_0 \supset G_1 \supset G_2 \supset \cdots$, $\bigcap_{n=0}^{\infty} G_n = \{0\}$, and the G_n's form a fundamental system of neighborhoods of zero in G. Furthermore, for each $n \geq 0$ there exists an $x_n \in G_n \backslash G_{n+1}$ such that $\chi_{m_n}(x_n) = \exp(2\pi i/p_{n+1})$, and each $x \in G$ can be represented uniquely by $x = \sum_{i=0}^{\infty} b_i x_i$, with $0 \leq b_i < p_{i+1}$ for all $i \geq 0$. Also

$$G_n = \left\{x \in G : x = \sum_{i=0}^{\infty} b_i x_i \,,\, b_0 = \cdots = b_{n-1} = 0\right\}.$$

Consequently, each coset of G_n in G can be represented as $z + G_n$, where $z = \sum_{i=0}^{n-1} b_i x_i$ for some choice of the b_i, $0 \leq b_i < p_{i+1}$. We shall denote these z, ordered lexicographically, by $z_{q,n}$, $0 \leq q < m_n$.

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