SOME IDENTITIES IN COMBINATORIAL ANALYSIS

BY L. CARLITZ

1. In a recent paper [3] Subbarao and Vidyasagar proved the following two identities.

$$(1.1)\qquad \sum_{n=0}^{\infty}(-1)^n a^n x^n (1+ax)(1+ax^3)(1+ax^5)\cdots(1+ax^{2n-1}) = 1+\sum_{n=1}^{\infty} a^{3n-1}x^{3n^2}(ax^{2n}-a^{-1}x^{-2n})$$

$$(1.2)\qquad \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n} x^{n(n+1)}}{(1+ax)(1+ax^3)(1+ax^5)\cdots(1+ax^{2n+1})} = 1+\sum_{n=1}^{\infty} a^{3n-1}x^{3n^2}(ax^{2n}-a^{-1}x^{-2n})$$

They also indicated the relationship of (1.1) and (1.2) to the following identity due to Watson [4].

$$(1.3)\qquad \prod_{n=1}^{\infty}\frac{(1-x^{2n})(1-a^2x^{2n-2})(1-a^{-2}x^{2n})}{(1+ax^{2n-1})(1+a^{-1}x^{2n-1})} = \sum_{n=-\infty}^{\infty}(a^{-3n}-a^{3n+2})x^{n(3n+2)}.$$

It is assumed as usual that $|x| < 1$.

In this note we give a simple proof of (1.1) and (1.2) that is reminiscent of some of the proofs of the Rogers–Ramanujan identities (see for example [1] and [2; Chapter 6]). We also give a simple proof of (1.3). Moreover we show that (1.3) can be obtained from the Jacobi theta formula.

2. We recall that

$$(2.1)\qquad (1+a)(1+ax)\cdots(1+ax^{n-1}) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} a^k x^{\frac{1}{2}k(k-1)},$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k+1})}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Replacing x by x^2 and a by ax in (2.1) we get

$$(2.2)\qquad (1+ax)(1+ax^3)\cdots(1+ax^{2n-1}) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}' a^k x^{k^2},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}'$ is obtained from $\begin{bmatrix} n \\ k \end{bmatrix}$ by replacing x by x^2.

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