AN ASYMPTOTIC ESTIMATE FOR A CLASS OF DIVISOR SUMS

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In a paper by Elliott and Halberstam [2], estimates for sums of the form

$$\sum_{a<p\le x}\ \sum_{n\mid(p-a)} f(n) \quad\text{and}\quad \sum_{p<n}\ \sum_{d\mid(n-p)} f(d)$$

are obtained, under the condition $f(x) \ll 1$. (The symbol $\ll$ denotes an inequality with an unspecified constant, and p always denotes a prime.) In this paper we consider the same sums for some functions f which do not satisfy $f(x) \ll 1$.

The principal result obtained is the following:

Theorem 1. *Let f be a differentiable function defined for $x > 0$, which satisfies:*

(i) $f(x\,y) = f(x)f(y)\left(1 + O\left(\dfrac{1}{\log^{\beta}(\max(x, y))}\right)\right)$

(ii) *if* $0 \le y < x$; $f(x \pm y) = f(x)\left(1 + O\left(\dfrac{y}{\log^{\beta} x}\right)\right)$

(iii) $\dfrac{f(x)}{\log x}$ *is non-decreasing for x sufficiently large*

(iv) $f(x) \le x^{\gamma}$ *for $x > 0$ and* $x^{\gamma/2+\delta} \le f(x)$ *for* $x > 1$,

where β, γ, δ are positive constants, $\beta \ge 2$, $\delta < \gamma/2$. Then

$$S = \sum_{a<p\le x}\ \sum_{n\mid(p-a)} f(n) = \sum_{\substack{j=1\\(a,j)=1}}^{\infty} \frac{f\left(\frac{1}{j}\right)}{\varphi(j)} \int_2^x \frac{f(t)}{\log t}\,dt + O\left(\frac{xf(x)}{\log^{\beta} x}\right),$$

where (m, n) denotes the greatest common divisor of m and n, and $\varphi(m)$ is the number of positive integers less than m which are relatively prime to m.

The conditions on f are not the weakest possible, but suffice for the most interesting applications. (See Corollary 1.)

Proof.

$$\begin{aligned} S &= \sum_{a<p\le x}\ \sum_{n\mid(p-a)} f(n) \\ &= \sum_{n\le\sqrt{x-a}}\ \sum_{\substack{j\le(x-a)/j\\ nj=p-a}} f(n) + \sum_{j\le\sqrt{x-a}}\ \sum_{\substack{\sqrt{x-a}<n\le(x-a)/j\\ nj=p-a}} f(n). \end{aligned}$$

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