A SORTING FUNCTION

By L. CARLITZ

To John H. Roberts
on his sixty-fifth birthday

1. Morris [1] has defined a function $F(n)$ in the following way. Suppose that $n - 1$ items $A_1, \cdots, A_{n-1}$ have been sorted into linear order and that another item A_n must be inserted in its proper order. Then $F(n)$ is defined as the average number of comparisons required to insert the nth item. Then it is shown, to begin with, that

$$F(1) = 0,$$

$$F(n) = 1 + \min_{1 \le k < n} \left\{ \frac{k}{n} F(k) + \frac{n-k}{n} F(n-k) \right\}.$$

If we put

$$G(n) = nF(n), \tag{1.1}$$

it is proved that $G(n)$ can be defined recursively by $G(1) = 0$ and

$$G(n) = n + G(m) + G(n - m), \tag{1.2}$$

where $m = [n/2]$, the greatest integer $\le n/2$. Morris shows also that

$$G(2^k) = k \cdot 2^k \tag{1.3}$$

and that $G(n)$ is linear in any interval $2^k \le n \le 2^{k+1}$; moreover

$$G(n) \ge n \log_2 n. \tag{1.4}$$

2. We shall now obtain an explicit formula for $G(n)$, namely

$$G(2^k + j) = k \cdot 2^k + j(k + 2) \quad (0 \le j \le 2^k). \tag{2.1}$$

It is convenient to define $G(0) = 0$. Then (1.2) becomes

$$\begin{cases} G(2n) = 2n + 2G(n) \\ G(2n + 1) = 2n + 1 + G(n) + G(n + 1); \end{cases} \tag{2.2}$$

the first of (2.2) holds for $n \ge 0$, while the second holds for $n \ge 1$. Thus to prove (2.1) it suffices to prove

$$G(2^k + 2j) = k \cdot 2^k + 2j(k + 2) \quad (0 \le j \le 2^{k-1}) \tag{2.3}$$

and

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