EXISTENCE OF TOPOLOGIES FOR COMMUTATIVE RINGS WITH IDENTITY

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Introduction. We shall prove that every infinite ring (≡ commutative ring with 1) has a proper (≡ separated nondiscrete ring) topology. The proof depends on the following result:

Proposition 1. *Every infinite ring A satisfies at least one of the following conditions:*

a) *A has a proper topology with a family of ideals as a neighborhood basis at* 0 (*an ideal topology*).

b) *A contains infinitely many nilpotents.*

c) *There is an element $a \in A$ such that $A/Ann_A a$ is an infinite field.*

Case b) is handled in §2, while in case c) the result can be deduced from the results in [4] by a trick.

Conversations with John O. Kiltinen were instrumental in shaping this paper. The main result rests equally on his work in [4] and my work here and in [2]; it was announced jointly [3].

1. Proof of Proposition 1. Assume that A is an infinite ring which fails to satisfy all three of the conditions a), b), and c). We shall obtain a contradiction. By [2; Theorem 2], A is a finite product of indecomposable rings having no ideal topology. At least one of these, call it B, is infinite. All three conditions must also fail for B. That b) fails is obvious. Suppose that there is an element $b \in B$ such that $B/Ann_B b$ is an infinite field. We have $A = B \times C$ for some C. Let $a = (b, 0)$. Then clearly, $A/Ann_A a \cong B/Ann_B b$, a contradiction. Thus, we may assume without loss of generality that A is indecomposable, and this means that the only idempotents in A are 0 and 1.

Since A has no ideal topology, we know from Theorem 1 of [2] that there are only finitely many maximal ideals $M_1, \cdots, M_k$ of A having nonzero annihilator, and that each nonzero element of A has a nonzero multiple in $U = \bigcup_{i=1}^{k} Ann_A M_i$. We shall show that $U \subset N$, the radical of A (by the radical of a ring we always mean the ideal of all nilpotent elements), and hence that U is finite. Suppose $a \in Ann_A M$, where M is one of the M_i. To show $a \in N$, it suffices to show that $a \in M$, for then $a^2 = 0$. But if $a \notin M$, then $A/Ann_A a = A/M$ is a field, hence a finite field, and a represents a nonzero residue class. Then $a^h \equiv 1$ modulo M for some positive integer h, and we have $a^h = 1 + b$, $b \in M$. Since $aM = 0$, $a^{2h} = a^h + a^h b = a^h$, so that a^h is idempotent, and a^h

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