TOPOLOGICAL DEGREE AND THE NUMBER OF SOLUTIONS OF EQUATIONS

By Jane Cronin

1. Introduction. In [3], some generalizations of aspects of the Fundamental Theorem of Algebra were obtained for operator equations of the form

$$(I - P)x = y$$

where I is the identity and P is a completely continuous (compact) polynomial operator. Here we obtain results of this kind for a wider class of operator equations: first for operators of the form $I - C$ where I is the identity and C is a compact continuously differentiable operator; second for operators for which the Browder degree $\deg_1$ (see Browder [1]) is defined. As in [3], the underlying idea is to look at an equation in a real Banach space B and at its extension to an equation in the complex Banach space $B \times B$. It is shown that the number t of solutions of the equation in B is bounded by the topological degree d of the operator in the corresponding equation in $B \times B$ and that $t \equiv d \pmod 2$. Also, if d_0 is the topological degree of the operator in the equation in B, then $t \geq d_0$ and $d_0 \equiv d \pmod 2$. These generalizations of [3] include a much wider class of mappings than those in [3], and the proofs are simpler than the proofs in [3]. However, in obtaining the more general results, we lose some precision: for polynomial operators, we can make an explicit computation of the topological degree or an upper bound for the degree (see [3, Theorem 1]); such a computation is not generally possible for the wider class of mappings studied here.

Following Browder [1], we denote the Leray-Schauder degree at q of a mapping $I - C$ relative to a set $\bar{G}$ which is the closure of a bounded open set by $\deg_{LS}(I - C, \bar{G}, q)$. The Browder degree of a mapping f, which can be represented in the form $h - C$ where h is a homeomorphism and C is compact, is denoted by $\deg_1([f, h], \bar{G}, p)$. We also use $\deg(F, \bar{G}, q)$ to denote the Brouwer degree at q of a map F (in Euclidean space) relative to $\bar{G}$, where G is a bounded open set in Euclidean space.

2. The theorem for Leray-Schauder degree. Let B be a real Banach space and $\mathcal{B} = B \times B$ the underlying real linear space for the complexification of B. For $w \in B \times B$ let w^* denote the conjugate of w, i.e., if $w = (x, y)$, let $w^* = (x, -y)$. (We will also use * to denote the conjugate of a point in complex Euclidean n-space.) Let C be a compact map of B into itself such that C is

Received September 12, 1969. The research in this paper was supported partly by the U. S. Army Research Office (Durham) (DA-ARO-D-31-124-G1098) and partly by a Rutgers Research Council Faculty Fellowship.