TRACE FUNCTIONS, I

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1. Introduction. It is the purpose of this paper to introduce a natural generalization of the trace of a matrix. An application to partitioned hermitian matrices is given.

2. A generalization of trace. Let F be the field of complex numbers. Let V be a unitary space over F of dimension n. Denote by $M_n(F)$ the set of $n \times n$ matrices over F. Let G be a subgroup of order r of the symmetric group S_n and suppose $\chi : G \to F$ is a character of degree m.

Define $T_\chi^G : M_n(F) \to F$ as follows: For any $A = (a_{ij})$ in $M_n(F)$,

$$T_\chi^G(A) = \sum_{\sigma \varepsilon G} \chi(\sigma) \sum_{t=1}^{n} a_{t\sigma(t)} .$$

If G is the singleton group and χ is of degree one, then T_χ^G is the trace. We list some trivial facts as lemmas.

2.1 LEMMA. *T_χ^G is a linear functional on $M_n(F)$.*

2.2 LEMMA. *Let $\bar{\chi}(\sigma) = \overline{\chi(\sigma)}$; then*

$$T_\chi^G(A^T) = T_{\bar{\chi}}^G(A) = \overline{T_\chi^G(\bar{A})}, \quad \textit{and} \quad T_\chi^G(A^*) = \overline{T_\chi^G(A)};$$

(A^T is the transpose of A, $\bar{A} = (\bar{a}_{ij})$, and $A^* = \overline{A^T}$).

2.3 LEMMA. *$T_\chi^G(B^*A) = \overline{T_\chi^G(A^*B)}$.*

Let $f : M_n(F) \to F$ be defined as follows: $f(A) = \sum_{i,j=1}^{n} a_{ij}$.

2.4 LEMMA. *If $H \varepsilon M_n(F)$ is positive semidefinite (positive definite) hermitian* [2], *then $f(H) \geq 0$ ($f(H) > 0$).*

Proof. Let $e = (1, \cdots, 1)$. Then $eHe^T = f(H)$.

When $\sigma \varepsilon S_n$, let $P(\sigma) \varepsilon M_n(F)$ be the matrix whose i, j-th element is $\delta_{i,\sigma(i)}$. Let $C(\chi, G) = \sum_{\sigma \varepsilon G} \chi(\sigma) P(\sigma)$.

2.5 LEMMA. *$C(\chi, G) = C(\chi, G)^*$.*

Proof. $C(\chi, G)^* = \sum_{\sigma \varepsilon G} \overline{\chi(\sigma)}\, P(\sigma)^T = \sum_{\sigma \varepsilon G} \chi(\sigma^{-1}) P(\sigma^{-1})$.

2.6 LEMMA. *If χ is irreducible, then $C(\chi, G)^2 = (r/m) C(\chi, G)$.*

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