ON THE CONVERGENCE OF THE ZETA FUNCTION OF A COMMUTATIVE RING

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1. Introduction. It is well known [3] that if X is a scheme of finite type, the zeta function of X has abscissa of convergence dim X and can be extended as a meromorphic function to dim $X - \frac{1}{2} < \text{Re}(s)$ with a simple pole at dim X. In this paper we shall show that given any positive prime p (or 0) and any real number $\alpha \geq 0$, there is an integral domain of characteristic p (or 0) whose zeta function has abscissa of convergence α and cannot be extended as a meromorphic function past α.

2. The main theorem. The principal result of this paper is

2.1 Theorem. *Let p be a positive prime (or 0) and $\alpha \geq 0$ be a real number. Then there is an integral domain A of characteristic p (or 0) such that the abscissa of convergence of $\zeta_A(s)$ is α, and for all $r \in Z^+$ $\lim_{s\to\alpha^+} (s - \alpha)^r \zeta_A(s) = \infty$.*

The idea behind the proof is to eliminate certain factors from the zeta function of a suitable domain B to obtain a function $\zeta(s)$ such that for some $n \geq 0$ the function $\zeta(s + n)$ has the properties of $\zeta_A(s)$ in Theorem 2.1, and then to show that $\zeta(s + n)$ is the zeta function of a domain A. The domain A will be a polynomial ring over B in n indeterminants localized at a multiplicative subset of B.

3. Preliminary results. Throughout this paper A and B will denote commutative rings with unity. Spm A will denote the set of maximal ideals of A.

Suppose that A is a commutative ring, and $A/\mathfrak{M}$ is a finite field for each $\mathfrak{M} \in \text{Spm } A$. Then the zeta function of A is defined by

$$\zeta_A(s) = \prod_{\mathfrak{M} \in \text{Spm } A} [1 - N(\mathfrak{M})^{-s}]^{-1},$$

where s is a complex number and $N(\mathfrak{M})$, the norm of $\mathfrak{M}$, is the number of elements in the finite field $A/\mathfrak{M}$. Assume that $\zeta_A(s)$ converges for some s. Then we can order Spm A (for instance by increasing norm) and write $\zeta_A(s) = \prod a_n(s)$, where $a_n(s) = [1 - N(\mathfrak{M})^{-s}]^{-1}$ for some $\mathfrak{M} \in \text{Spm } A$. In this case, if s, t are positive numbers, observe that

3.1 $$0 < \log a_n(t) \leq \log a_n(s) \text{ if } 0 < s \leq t.$$

If Spm A is infinite, then

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