A CUT POINT THEOREM FOR PLANE CONTINUA

By Charles L. Hagopian

Let M be a compact plane continuum and let y be a point of M. If there exist distinct points x and z in $M - \{y\}$ such that each subcontinuum of M that contains x and z contains y, then y *cuts* M between x and z, and y is said to be a *cut point* of M. If M is totally nonaposyndetic, then M has a cut point [3, Theorem 12]. However, a compact plane continuum may be nonaposyndetic at some of its points and not have a cut point. For each point x of M, F. B. Jones defines K_x to be the set consisting of x and all points y in $M - \{x\}$ such that M is not aposyndetic at x with respect to y. The set K_x is closed [3, Theorem 2], but it need not be a continuum [1, Example 4]. In this article the existence of a cut point of M is established when for some point x of M the set K_x is not connected. In fact, it is shown that each point of K_x − (x-component of K_x) is a cut point of M.

Throughout this paper S is the set of points of a simple closed surface (that is, a 2-sphere). For definitions of unfamiliar terms and phrases see [4].

Definition 1. Let M be a continuum in S and let x and y be distinct points of M. The set $S - M$ is said to be *improperly folded* around x with respect to y if there exist two monotone descending sequences of circular regions $U_1, U_2, U_3, \cdots$ and $V_1, V_2, V_3, \cdots$ in S centered on and converging to x and y respectively such that $\mathrm{Cl}\, U_1 \cap \mathrm{Cl}\, V_1 = \varnothing$ ($\mathrm{Cl}\, U_1$ is the closure of U_1), and there exists a sequence of mutually exclusive sets $X_1, X_2, X_3, \cdots$ in $S - M$ having the following properties. For each positive integer i, the set X_i is the union of two intersecting arc-segments (open arcs) I_i and T_i such that

(1) $I_i \cap T_i$ is connected

(2) I_i is contained in $\mathrm{Bd}\, U_i$ ($\mathrm{Bd}\, U_i$ is the boundary of U_i) and has endpoints a_i and b_i in M

(3) the sets $\mathrm{Cl}\, U_{i+1}$ and (a_i-component of $M - V_i$) $\cup$ (b_i-component of $M - V_i$) are disjoint

(4) T_i is contained in $S - \mathrm{Cl}\,(V_i \cup U_{i+1})$ and has two distinct endpoints in $\mathrm{Bd}\, V_i$

(5) $T_i \cup \mathrm{Bd}\, V_i$ contains a simple closed curve S_i that separates a_i from $\{y, b_i\} \cup \mathrm{Cl}\, U_{i+1}$ in S

(6) for each positive integer j not equal to i, the simple closed curve S_j does not separate the point x from a_i in S.

When the sequences can be found to satisfy conditions (1)–(5), $S - M$ is said to be *folded* around x with respect to y. For distinct points x and y of M, the

Received July 30, 1969. Revised July, 1971. Supported by the Air Force Office of Scientific Research.