GROUP EXTENSIONS OF NULL SEMIGROUPS

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In this paper we find all ideal extensions of any null semigroup by a group with zero; we refer to these as *group extensions*. Our findings are then applied to characterize a class of subdirectly irreducible, commutative semigroups.

As is shown in a result of R. Yoshida [12], the ideal extension of any semigroup by a second semigroup is closely associated with the translational hull of the former. This association was demonstrated for weakly reductive semigroups by A. H. Clifford [3]. We shall describe the translational hull of any null semigroup, along with presenting the maximal subgroups of this translational hull in §2. This facilitates in §3 a determination of the structure of all group extensions of any null semigroup modulo certain group homomorphisms.

The main goal of this paper, viz., the description of all finite commutative, subdirectly irreducible semigroups which are group extensions of null semigroups, is presented in §4. Our work in this section is motivated by the results of B. M. Schein [7], [8], and our result solves a special case of an unsolved problem emanating from the latter's work.

1. Preliminaries. Let S be a semigroup. A transformation λ of S, written as a left operator, is a *left translation* of S if $\lambda(xy) = (\lambda x)y$ for all x, y in S; a transformation ρ of S, written as a right operator, is a *right translation* of S if $(xy)\rho = x(y\rho)$ for all x, y in S. A pair (λ, ρ) consisting of a left translation λ and a right translation ρ with the property $x(\lambda y) = (x\rho)y$ for all x, y in S is called a *bitranslation* of S. The collections $\Lambda(S)$, $P(S)$ of all left translations, right translations of S are semigroups under the operation of mapping composition; the collection $\Omega(S)$ of all bitranslations of S with multiplication induced by the direct product $\Lambda(S) \times P(S)$ is a semigroup called the *translational hull* of S.

Let X be a nonempty set. A function α mapping a subset Y of X into X is called a *partial transformation* of X. The set Y is called the *domain* of α and is denoted by $\mathbf{d}\alpha$; the set of all x in X such that $\alpha y = x$ for some y in Y is called the *range* of α and is denoted by $\mathbf{r}\alpha$. For convenience, the *empty transformation* $\varnothing$ is defined to be the mapping of the empty set $\varnothing$ into X (onto $\varnothing$), so that $\mathbf{d}\varnothing = \mathbf{r}\varnothing = \varnothing$.

Composition of partial transformations α, β of X is defined by $(\alpha\beta)x = \alpha(\beta x)$ for all x such that x is in $\mathbf{d}\beta$ and βx is in $\mathbf{d}\alpha$; otherwise, $(\alpha\beta)x$ is not defined. Under this composition, the set of all partial transformations of X is a semigroup, which we denote by $W(X)$. Note that $W(X)$ has the element $\varnothing$ as its zero. The semigroup of all partial transformations (under composition) written as

Received July 31, 1969. This paper is based on a part of the author's Ph.D. dissertation at The Pennsylvania State University; the author wishes to thank his advisor, M. Petrich.