A PICONE IDENTITY FOR STRONGLY ELLIPTIC SYSTEMS

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1. **Introduction.** The Picone identity is usually associated with functions $u(x)$ and $v(x)$ which are, respectively, solutions of Sturm–Liouville equations of the form

$$(1.1) \qquad (au')' + cu = 0,$$

$$(1.2) \qquad (gv')' + hv = 0.$$

If $v(x) \neq 0$ in $[x_1, x_2]$, then

$$\frac{d}{dx}\left[\frac{u}{v}(au'v - guv')\right] = u(au')' - \frac{u^2}{v}(av')' + (a - g)u'^2 + g\left(u' - \frac{u}{v}v'\right)^2.$$

Making use of the differential equations (1.1) and (1.2) one obtains

$$(1.3) \qquad \frac{d}{dx}\left[\frac{u}{v}(vau' - ugv')\right] = (h - c)u^2 + (a - g)u'^2 + g\left(u' - \frac{u}{v}v'\right)^2.$$

Integrating both sides of (1.3) from x_1 to x_2 one obtains Picone's identity and a direct proof of the Sturm–Picone comparison theorem [4; 228].

The identity (1.3) has been generalized to M independent variables by Picone [8] and the author [5] in order to establish Sturmian comparison theorems for solutions of selfadjoint elliptic equations of the form

$$(1.4) \qquad \sum_{i,j=1}^{M} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu = 0,$$

$$(1.5) \qquad \sum_{i,j=1}^{M} \frac{\partial}{\partial x_i}\left(g_{ij}\frac{\partial v}{\partial x_j}\right) + hv = 0.$$

If $u(x_1, \cdots, x_M)$ and $v(x_1, \cdots, x_M)$ satisfy (1.4) and (1.5) in a domain $D \subset R^M$ and if $v(x) \neq 0$ in $\bar{D}$, then (1.3) generalizes to

$$(1.6) \qquad \sum_i \frac{\partial}{\partial x_i}\left[\frac{u}{v}\left(v\sum_j a_{ij}\frac{\partial u}{\partial x_j} - u\sum_j g_{ij}\frac{\partial v}{\partial x_j}\right)\right] = (h - c)u^2$$
$$+ \sum_{i,j}(a_{ij} - g_{ij})\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j} g_{ij}\left(\frac{\partial u}{\partial x_i} - \frac{u}{v}\frac{\partial v}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_j} - \frac{u}{v}\frac{\partial v}{\partial x_j}\right).$$

Integrating over D and applying Green's Theorem one obtains a direct proof of the Sturm–Picone Theorem for elliptic equations.

In the present paper we shall study selfadjoint strongly elliptic systems of