PAIRS OF RELATIVELY PRIME INTEGERS IN RESIDUE CLASSES AND BINARY COMPOSITIONS

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1. Introduction. Letting b_1, b_2, r denote integers, $r > 0$, and x, y real numbers, $y \geq x \geq 1$, we define $U_r(x, y) = U_r(x, y; b_1, b_2)$ to be the number of pairs $\{a_1, a_2\}$ of positive, relatively prime integers, a_1, a_2, such that $a_1 \leq x$, $a_2 \leq y$, and $a_1 \equiv b_1 \pmod{r}$, $a_2 \equiv b_2 \pmod{r}$. It was proved in [2] that if $(b_1, b_2, r) = 1$, then

$$U_r(x, y) = 6xy/\pi^2 J(r) + O(y \log x/r) + O(x), \tag{1.1}$$

uniformly in b_1, b_2, where $J(r)$ is the Jordan totient function of order 2. Evidently, if $(b_1, b_2, r) \neq 1$, then $U_r(x, y) = 0$ for all x, y.

For positive integers m, n, $m \leq n$, define $Q(m, n)$ to be the number of ordered sets of positive integers, x_1, x_2, y_1, y_2, such that

$$\{m, n\} = \{x_1, x_2\} + \{y_1, y_2\}, \qquad (x_1, x_2) = (y_1, y_2) = 1. \tag{1.2}$$

In the paper [2] it was proved as an application of (1.1) that

$$Q(m, n) = \alpha(m, n)mn + \begin{cases} O(n \log^2 m) & \text{if } n \geq m^2 \\ O(m\sqrt{n} \log^2 m) & \text{if } n \leq m^2, \end{cases} \tag{1.3}$$

where $\alpha(m, n)$ is a function whose values are contained between positive bounds. In the case $n \geq m^2$, this result represents an improvement by a logarithmic factor over an estimate proved by a different method in an earlier paper [1]. It had also been proved in [1] by the same method that for all $\epsilon > 0$,

$$Q(m, n) = \alpha(m, n)mn + O((mn)^{\frac{1}{2}+\epsilon}) \tag{1.4}$$

if $n \leq m^2$. Thus, in general, the estimate (1.3) in case $n \leq m^2$ is inferior to the previous result (1.4).

In the present paper we prove the refinement of (1.1) contained in Theorem 1 (§2). This result is applied to §3 to prove an improved estimate for $Q(m, n)$ in case $n \leq m^{2-a}$, where $a > 0$ (Theorem 2). In particular, it is shown that in this case (1.4) is true with $\epsilon = 0$. In §4 we obtain, on the basis of (1.1), bounds for the smallest relatively prime pair of integers in two residue classes (mod r).

The question of the density of relatively prime integers in congruence classes is considered in §5.

The methods of the paper are entirely elementary. In §§2 and 3 we assume some of the preliminary formulas proved in [2].

Received July 22, 1969. This research was partially supported by a grant from the National Science Foundation (G-19012).