INJECTIVE AND PROJECTIVE STONE ALGEBRAS

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1. Introduction. A Stone algebra $\langle S; \wedge, \vee, {}^*, 0, 1\rangle$ or, simply, S, is an algebra with two binary $(\wedge, \vee)$, one unary $({}^*)$ and two nullary $(0, 1)$ operations satisfying the following axioms: (i) $\langle S; \wedge, \vee\rangle$ is a distributive lattice; (ii) $0 \vee x = x \wedge 1 = x$; (iii) $a \wedge x = 0$ iff $x \leq a^*$; (iv) $a^* \vee a^{**} = 1$. In other words, a Stone algebra is a pseudo-complemented distributive lattice satisfying (iv).

In this paper we give a complete characterization of injective Stone algebras, and of finite projective Stone algebras. The characterization problem of infinite projective Stone algebras is left open.

§2 contains the background material on Stone algebras. In §3 we make a few categorical observations of a trivial, but useful, nature and recall some theorems about the category of distributive lattices. It is proved in §4 that $\mathfrak{S}_3$ (the three element Stone algebra) is injective; hence every Stone algebra can be embedded in an injective one, and a few further properties of injective Stone algebras are derived, in particular, they are proved to be representable over their center. Stone algebras having a full representation over their center are discussed in §5, resulting in the characterization theorem of injective Stone algebras. The characterization problem of finite projective Stone algebras is settled in §6.

2. Stone algebras. Let $\mathfrak{S}_2$ and $\mathfrak{S}_3$ denote the two and three element Stone algebras; the only elements of $\mathfrak{S}_2$ are 0, 1; the elements of $\mathfrak{S}_3$ are 0, a, 1.

Theorem A (G. Grätzer [5]). *Every Stone algebra is isomorphic to a subalgebra of* $\mathfrak{S}_2^n \times \mathfrak{S}_3^m$.

For a Stone algebra S set

$$C(S) = \{a^* \mid a \in S\},$$

$$D(S) = \{a \mid a^* = 0, a \in S\}.$$

$C(S)$ is the set of complemented elements of S and it is called the *center* of S; $D(S)$ is the *dense set* of S. $C(S)$ is a Boolean algebra, in fact a subalgebra of S. The map $\varphi : x \rightarrow x^{**}$ is a homomorphism of S onto $C(S)$. In fact, φ is a *retraction*, that is, an idempotent endomorphism, and so $C(S)$ is a *retract* of S. $D(S)$ is a dual ideal of S; in particular, it is a distributive lattice with 1.

Theorem B. *Let P be a prime ideal in a Stone algebra S and $a, b \in S$.*

(i) $(a \vee b)^* = a^* \wedge b^*$, $(a \wedge b)^* = a^* \vee b^*$.

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