ON CERTAIN IDEALS OF $C(X)$

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Throughout this paper we let X denote a completely regular Hausdorff space. By $C(X)$ is meant the ring of real-valued continuous functions on X with the pointwise operations. If $p \in \beta X$, then M_p is the maximal ideal in $C(X)$ associated with p and O_p the intersection of the prime ideals of $C(X)$ contained in M_p. In the first part of this note we characterize the maximal ideals of M_p and O_p (as rings in their own right). We then prove the following extension theorem. If I and J are ideals of $C(X)$ contained in unique real maximal ideals, then any ring isomorphism ϕ of I onto J can be extended to a ring automorphism of $C(X)$ onto itself. This automorphism is, of course, induced by a homeomorphism of vX onto itself. If X is realcompact, then X is homogeneous if and only if every pair of real maximal ideals are ring isomorphic. We also give (apparently) stronger sufficient conditions for the homogeneity of a realcompact space.

In what follows let $\mathbf{Z}$ denote the integers and $\mathbf{R}$ the real numbers. Terms not defined in this note and uncited theorems can be founded in the Gillman and Jerison text [1].

1. The maximal ideals of maximal ideals of $C(X)$. If $f \in C(X)$, then $Z(f)$ denotes the set of zeros of f. An ideal J of $C(X)$ is termed a *z-ideal* provided that $f \in J$ and $Z(g) = Z(f)$ implies that $g \in J$.

Theorem 1.1. *Let J be a z-ideal of $C(X)$ and let I be a maximal ideal of J (as a ring). Then I is a prime ideal of J and an ideal of $C(X)$.*

Proof. If I is not a prime ideal of J, then there are elements $a, b \in J \sim I$ with $ab \in I$. By the maximality of I we have

$$J = \{na + pa + i : n \in \mathbf{Z}, p \in J, i \in I\}$$
$$= \{nb + pb + i : n \in \mathbf{Z}, p \in J, i \in I\}.$$

Let $c, d \in J$. Then for proper choices of $n, n' \in \mathbf{Z}$, $p, p' \in J$, and $i, i' \in I$ we have that $c = na + pa + i$ and $d = n'b + p'b + i'$. Then $cd = nn'ab + (np' + n'p + pp')ab + (n'b + p'b + i')i + (na + pa)i'$. Thus $cd \in I$. That is, $J^2 \subset I$. Let $f \in J$. Since J is a z-ideal, one can easily see that $f = (\sqrt{|f|})((\text{sign } f)\sqrt{|f|})$ where $\text{sign } f = f/|f|$ whenever $f(x) \neq 0$; thus $f \in J^2$. Therefore $J \subset J^2 \subset I$, a contradiction. Therefore I is a prime ideal of J. Now let $f \in C(X)$ and $p \in I$. Then $f^2p \in J$, so $(f^2p)p \in I$. Since I is prime, we get $fp \in I$.

Now let M and M' be distinct maximal ideals of $C(X)$. Since $M + M' = C(X)$, the "second isomorphism theorem" gives $M/M \cap M' \simeq C(X)/M'$.

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