THE DIAGONAL OF A DOUBLE POWER SERIES

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1. Synopsis. The *diagonal sequence* of a double sequence $\{f(m, n): m, n = 0, 1, \cdots\}$ is defined to be the sequence $\{f(n) = f(n, n): n = 0, 1, \cdots\}$. In this note we are interested in finding the diagonal sequence of a double sequence that has been defined by means of a recurrence relation without calculating all of the terms of the double sequence. We use generating functions; the main result is an integral representation for the diagonal of a double power series which represents an analytic function.

2. Introduction. We begin with an example. Let $\{f(m, n): m, n = 0, 1, \cdots\}$ denote the double sequence of integers defined by the linear homogeneous difference equation

$$f(m, n) = f(m, n - 1) + f(m - 1, n), \tag{1}$$

for $m, n = 1, 2, \cdots$, along with the initial conditions $f(m, 0) = f(0, n) = 1$ for $m, n = 0, 1, \cdots$.
These numbers have as their generating function

$$F(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n)x^m y^n, \tag{2}$$

and this definition together with (1) implies

$$F(x, y) = yF(x, y) + xF(x, y) + 1, \tag{3}$$

from which it follows that

$$F(x, y) = (1 - x - y)^{-1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{m} x^m y^n, \tag{4}$$

so $f(m, n) = \binom{m+n}{n}$ for $m, n = 0, 1, \cdots$. We set $f(n) = f(n, n)$, and call $\{f(n): n = 0, 1, \cdots\}$ the *diagonal* of the double sequence $\{f(m, n): m, n = 0, 1, \cdots\}$; the generating function of $\{f(n): n = 0, 1, \cdots\}$ is defined to be

$$F(x) = \sum_{n=0}^{\infty} f(n)x^n. \tag{5}$$

For example, when $f(m, n) = \binom{m+n}{n}$, we have

$$F(z) = \sum_{n=0}^{\infty} \binom{2n}{n} z^n = (1 - 4z)^{-\frac{1}{2}}; \tag{6}$$

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