WEAK SEMICOMPLEXES AND THE FIXED POINT THEORY OF TREE-LIKE CONTINUA

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In this paper we consider a regularity condition for tree-like continua and show that this condition completely characterizes those tree-like continua which admit weak semicomplex structures. This provides a purely topological description of the largest class of such continua which can be shown to have the fixed point property by existing algebraic methods.

1. Introduction and definitions. The term continuum will always stand for a compact, connected, Hausdorff space, and we shall use $\sum (X)$ for the collection of all finite covers of a space X by open sets. A tree is a one-dimensional, acyclic, finite simplicial complex. If X is a continuum and α is a finite collection of open subsets of X, then α is called a simple chain or a tree chain, according as the nerve of α, N_α, is an arc or a tree. A continuum X is called chainable, or tree-like, if there is a cofinal subfamily $\sum_0 (X)$ of $\sum (X)$ such that each member of $\sum_0 (X)$ is, respectively, a simple chain or a tree chain. These spaces have been discussed by Bing in [1], where metric chainable continua are referred to as snake-like.

Recall that a space X is said to have the fixed point property if every map of X into itself leaves some point fixed. Bing has called the question of whether or not all tree-like continua have the fixed point property one of the most interesting unsolved problems in geometric topology [3; 122]. A discussion of the history of this question and a guide to its literature can be found in [16, Chapter II].

Our interest in the present paper is in the algebraic approach of applying the Lefschetz fixed point theorem to tree-like continua. We will say that a space satisfies this theorem if every fixed point free self-map has a Lefschetz number of zero. Since tree-like continua are acyclic, all of their self-maps have Lefschetz number of one. Hence such spaces have the fixed point property if they satisfy the Lefschetz fixed point theorem. Dyer established this fact for chainable continua in [5] by showing that such spaces are quasi-complexes, as defined in [11; 322]. However, an example given by Chamberlin in [4] shows that not all tree-like continua are quasi-complexes.

Since other general settings for the Lefschetz fixed point theorem have been developed, it is natural to ask which tree-like continua can now be treated in this fashion. In particular, we note the weak semicomplexes defined by this author in [12] and [13], and the Q-simplicial spaces of Knill in [10]. Surprisingly,

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