A CHARACTERIZATION OF COMPLEMENTING SETS OF PAIRS OF INTEGERS

By Ivan Niven

1. Introduction. Let N be the set of non-negative integers, and let A, B, C be subsets of $N \times N$. If for every element $(c_1, c_2) \varepsilon C$ there exist elements $(a_1, a_2) \varepsilon A$ and $(b_1, b_2) \varepsilon B$ such that $(a_1, a_2) + (b_1, b_2) = (c_1, c_2)$, we write $A + B = C$. If in addition the elements (a_1, a_2) and (b_1, b_2) are unique we write $A + B \cong C$. If $A + B \cong N \times N$, we say that A, B is a complementing pair of sets in the plane. The purpose of this paper is to characterize all such pairs.

Rodney T. Hansen [1] has done recent work on related questions; in his §3 he remarks on the problem of characterizing all sets A, B such that $A + B \cong N \times N$. We follow the notation and definitions of Hansen to a considerable extent, with some deviation in detail. For example, the sets A, B form a complementing pair in Hansen's work if there is some subset C of $N \times N$ such that $A + B \cong C$; in the present paper a complementing pair A, B is one satisfying $A + B \cong N \times N$.

For any subset A of $N \times N$ define

$$A' = \{(a, 0) \mid (a, 0) \varepsilon A\}, \qquad A'' = \{(0, a) \mid (0, a) \varepsilon A\}.$$

A set A is said to be *proper* if $A = A' + A''$, otherwise *improper*. This amounts to saying that A is proper provided $(a_1, a_2) \varepsilon A$ implies $(a_1, 0) \varepsilon A$ and $(0, a_2) \varepsilon A$, and conversely. If A, B is a complementing pair and A and B are proper, then A, B is said to be a proper complementing pair, otherwise an improper complementing pair. (It turns out that if A, B is an improper complementing pair then *both* A and B are improper sets.)

It is an easy matter to characterize all proper complementing pairs by use of the characterization of de Bruijn [2] in the one-dimensional case. This we do, but first for completeness we formulate the de Bruijn solution. For any subsets I_1 and I_2 of N write $I_1 + I_2 \cong N$ if given any $n \varepsilon N$ there exist unique elements $i_1 \varepsilon I_1$ and $i_2 \varepsilon I_2$ such that $i_1 + i_2 = n$. If $I_1 + I_2 \cong N$ then $0 \varepsilon I_1$, $0 \varepsilon I_2$, and with no loss of generality we presume that $1 \varepsilon I_1$. Let $g_1, g_2, g_3, \cdots$ be any finite or infinite sequence of integers > 1, and define $G_1 = g_1$, $G_2 = g_1 g_2$, $G_3 = g_1 g_2 g_3$, etc. Define

$$I_1 = \{x_0 + x_2 G_2 + x_4 G_4 + \cdots \mid 0 \le x_0 \le g_1 - 1, 0 \le x_2 < g_3 - 1, \quad 0 \le x_4 \le g_5 - 1, \cdots\} \tag{1}$$

Received April 28, 1969. Research supported by N.S.F. Grant #GP 6510.