AN INTEGRAL EQUATION WITH BESSEL FUNCTION KERNEL

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The convolution type of integral equation

$$g(x) = \int_0^x (x - y)^{\alpha/2} J_\alpha[2\sqrt{k(x - y)}]f(y)\, dy$$

occurs frequently in the solution of dual integral equations. A formal solution of this was given by Burlak [2] and later by Srivastav [9]. Bharatiya gave a set of sufficient conditions under which the inversion exists [1]. Here we give fairly simple necessary and sufficient conditions that the solution may be square integrable.

1. Let $\mathfrak{h}_\alpha f$ denote the Hankel transform of f in the form used by Tricomi, i.e.

$$\mathfrak{h}_\alpha f = \mathfrak{h}_\alpha\{f(y); x\} = \int_0^\infty J_\alpha[2\sqrt{xy}]f(y)\, dy.$$

It is well known that if f is in $L_2\ (0, \infty)$, then so is $\mathfrak{h}_\alpha f$ which exists in the mean square sense [5; 213]. Also let $L_2^{(\nu)}$ denote the set of functions defined by Erdélyi [3; 300]. If $\alpha > 0$, f is in $L_2^{(-\alpha)}$ provided that f is in $L_2\ (0, \infty)$ and $\phi(t)$, the Mellin transform of f is such that $|t|^\alpha\ \phi(t)$ is in $L_2\ (-\infty, \infty)$.

Our results can now be stated as follows:

Theorem A. *The solution f of the integral equation*

$$g(x) = \frac{d}{dx}\int_0^x J_0[2\sqrt{k(x - y)}]f(y)\, dy \tag{1}$$

belongs to $L_2\ (0, \infty)$ if and only if (i) *$g(x)$ is in $L_2\ (0, \infty)$ and* (ii) *$g_1(t) = \mathfrak{h}_0 g$ vanishes in $0 \le t < a$. Under these conditions*

$$||g|| = ||f|| \tag{2}$$

where $||\ \ ||$ stands for L_2 norm.

Theorem B. *The solution f of the integral equation*

$$h(x) = k^{(1-\alpha)/2}\int_0^x (x - y)^{(\alpha-1)/2} J_{\alpha-1}[2\sqrt{k(x - y)}]f(y)\, dy, \qquad \alpha > 0 \tag{3}$$

belongs to $L_2\ (0, \infty)$ if and only if (i) *$x^{-\alpha}h(x)$ is in $L_2^{(-\alpha)}$ and* (ii) *$h_1(t) = \mathfrak{h}_{2\alpha}\{x^{-\alpha}h(x), t\}$ vanishes in $0 \le t < a$. Under these conditions*

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