THE ESSENTIAL CLOSURE OF AN ARCHIMEDEAN LATTICE-ORDERED GROUP

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1. Introduction. Throughout this paper l-group will always denote an Archimedean lattice-ordered group, and we shall confine our attention to such groups. Suppose that G is an l-subgroup of an l-group H (that is, G is a subgroup and a sublattice of H). We say that G is *large* in H or that H is an *essential extension* of G if for each l-ideal $L \neq 0$ of H, $L \cap G \neq 0$, and G is *essentially closed* if it admits no proper essential extension. An essentially closed essential extension of G will be called an *essential closure* of G.

Pinsker [11] proved that in the category of complete vector lattices each G admits a unique essential closure, and Jakubik [8] showed that this essential closure can be constructed solely from the underlying lattice structure of G. In this paper we make use of Bernau's representation of an l-group G to prove that G admits a unique essential closure, and thus obtain a new proof of Pinsker's theorem. Moreover, we show that the essential extensions of G are exactly those extensions that preserve the Boolean algebra of all polars of G and so the essential closure is the largest such extension.

An l-group A has the *splitting property* if it is a cardinal summand of each l-group that contains it as an l-ideal (that is, $G = A \boxplus B$, where this is the direct sum and for $a \in A$ and $b \in B$, $a + b$ is positive if and only if both a and b are positive). We show that each essentially closed l-group has the splitting property.

An l-subgroup G of H is *dense* if $0 < h \in H$ implies $0 < g < h$ for some $g \in G$. Clearly a dense subgroup is large. We list some examples of dense or essential extensions.

1. The (Dedekind MacNeille) completion $G^\wedge$ of G.
2. The lateral completion G^l of G (see below).
3. The divisible hull G^d of G.

G is dense in both $G^\wedge$ and G^l and if $0 < h \in G^d$, then $nh \in G$ for some $n > 0$, so all three are essential extensions of G. It follows that an essentially closed l-group is complete, laterally complete and divisible, and in §3 we prove the converse and show that $((G^d)^\wedge)^l$ is the essential closure of G.

Since the lateral completion of an l-group G is not a well-known concept, we give the definition in full. A subset $\{a_\lambda : \lambda \in \Lambda\}$ of G is called *disjoint* if $a_\alpha \wedge a_\beta = 0$ for each pair $\alpha \neq \beta$, and G is said to be *laterally complete* if each disjoint subset has a least upper bound. In [3] it is shown that for each l-group G there exists a unique l-group G^l such that G is a dense l-subgroup of G^l, G^l is

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