A NOTE ON REGULAR AND ANTI-REGULAR (WEAKLY) BOREL MEASURES

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Introduction. In this note we tie up some loose ends left in [3] by showing more clearly the relationships between regular, anti-regular, and weakly anti-regular weakly Borel measures. This note most properly should be considered as an addendum to [3]; we shall use the same notation as in [3] and, in many cases, give specific references to [3] rather than try to restate the needed results.

Preliminaries. Let ν, μ be measures on a σ-ring $\mathcal{S}$. Following [2], we say that ν is *S-singular* (or *weakly singular*) with respect to μ, denoted $\nu S \mu$, if for each $E \in \mathcal{S}$ there is a set $F \in \mathcal{S}$ such that $F \subset E$, $\nu(F) = \nu(E)$, and $\mu(F) = 0$. (See [3, §1] for properties of S-singularity.) ν is *absolutely continuous* with respect to μ, denoted $\nu \ll \mu$, if $\nu(E) = 0$ whenever $E \in \mathcal{S}$ and $\mu(E) = 0$ [1; 124].

We assume throughout the remainder of this paper that X is a locally compact, Hausdorff topological space. We shall use $\mathcal{B}$, $\mathcal{B}_\delta$, $\mathcal{B}_w$, and $\mathcal{B}_{\delta w}$ to denote the σ-rings generated by the compact, compact G_δ, closed, and closed G_δ subsets of X, respectively. We shall call them the *Borel, Baire, weakly Borel* (w. B.) and *weakly Baire* (w. Ba.) subsets of X, respectively. Of course, $\mathcal{B}_\delta \subset \mathcal{B} \subset \mathcal{B}_w$ and $\mathcal{B}_\delta \subset \mathcal{B}_{\delta w} \subset \mathcal{B}_w$. *Measures defined on these classes of sets which are finite on compact sets will carry the same name.*

Let ν be a w. B. measure on X.

(I) ν is *regular* if
$$\nu(E) = L\,U\,B\{\nu(C); E \supset C \text{ compact}\} \text{ for all } E \in \mathcal{B}_w .$$
(II) ν is *anti-regular* if $\nu S \nu'$ where ν' denotes the unique regular w. B. measure which agrees with ν on $\mathcal{B}_\delta$ [3, Lemma 2.1]. (We call ν' the *regular relative* of ν.)

(III) ν is *weakly anti-regular* if $\lambda = 0$ is the only regular w. B. measure λ such that $\lambda \leq \nu$.

Every anti-regular w. B. measure is weakly anti-regular, but the converse fails [3, Theorem 5.1 and Example 3.5].

Main results. In [3, Theorem 5.3] it is shown that $\nu S \mu$ for every regular w. B. measure ν and every weakly anti-regular w. B. measure μ. Our first two results show that this condition characterizes both regular and weakly anti-regular w. B. measures.

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