POLYNOMIAL APPROXIMATION AND ANALYTIC STRUCTURE

By H. Alexander

We shall study here uniform algebras on the unit interval I and the circle T and the related question of uniform approximation by polynomials on curves in $\mathbb{C}^n$. The following result has been obtained by H. S. Shapiro and A. L. Shields [12].

Theorem. *Let f, g be complex-valued continuous functions on* [0, 1] *and suppose*

(i) *There exist a, b with $0 \leq a < b \leq 1$ such that $f(a) = f(b)$*
(ii) *For all s, t with $0 \leq s < t \leq 1$, $f(s) = f(t)$ implies $s = a$ and $t = b$.*
(iii) $g(a) \neq g(b)$;

then the uniform algebra generated by f and g is $C(I)$, the algebra of all continuous complex-valued functions on I.

The following theorem generalizes this result since f above is locally $1 - 1$.

Theorem 1. *Let A be a uniform algebra on I such that there is $f \in A$ with f locally $1 - 1$. Then $A = C(I)$.*

Another sort of generalization (take E to be a two-point set in the following to obtain the Shapiro–Shields result) is

Theorem 2. *Let A be a uniform algebra on I generated by two functions f and g (so in particular, together they separate the points of I) such that there is a totally disconnected compact subset E of I satisfying*

(i) *f/E is constant*
(ii) *f separates every pair of points of I not both of which are in E.*

Then $A = C(I)$.

The proofs of these theorems will depend on the notion of analytic structure. Results in this direction were first obtained by J. Wermer [15], [17], further developed by E. Bishop [2], [3] and H. Royden [9] and then by G. Stolzenberg [14] who proved a result which will be one of our basic tools:

Theorem. *Let $X \subseteq \mathbb{C}^n$ be a compact polynomially convex set. Let $K \subseteq \mathbb{C}^n$ be a finite union of smooth (i.e. $\mathcal{C}^1$) curves. Then $(X \cup K)\hat{} - X \cup K$ is a (possibly empty) 1-dimensional analytic subset of $\mathbb{C}^n - X \cup K$. (Here $(\ )\hat{}$ denotes the polynomially convex hull. For this and further notions otherwise unexplained here we refer to* [14], *whose definitions and notations we shall follow.)*

Theorems 1 and 2 assume no smoothness, but rather that one function does

Received March 28, 1969.