BOCHNER–RAIKOV THEOREM FOR A GENERALIZED POSITIVE DEFINITE FUNCTION

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1. In a recent paper [7] the author generalized the concept of positive definite function and showed that a continuous generalized positive definite function on a topological group G is also of the form $p(t) = (U_t f, f)$, where $t \to U_t$ is a continuous representation of the group G by generalized unitary operators U_t acting on some Hilbert module H and $f \in H$ ((,) is the generalized scalar product on H). He also generalized Stone's theorem (Theorem 3 in [7]) by showing that for each continuous generalized unitary representation $t \to U_t$ of a commutative group G there exists a generalized spectral measure $P: \Delta \to P_\Delta$ on $\hat{G}$ such that $U_t = \int_{\hat{G}} \overline{(t, \alpha)}\, dP_\alpha$ for all $t \in G$ ($\hat{G}$ here denotes the group of characters on G).

The present work deals with a generalization of Bochner's theorem [4,36A] which establishes a correspondence between positive definite functions on G and certain positive measures on G (see also the statement of Raikov's theorem in §31 of [5]). We shall show that each continuous generalized positive definite function (with values in some H^*-algebra $\mathfrak{A}$) is of the form $p(t) = \int_{\hat{G}} (t, \alpha) dm(\alpha)$ for some $\mathfrak{A}$-valued positive (in a certain sense) bounded measure m on $\hat{G}$. It turns out that the generalized Bochner theorem will be a simple consequence of the two above-mentioned results of [7] if one can show that the integral and the generalized scalar product can be interchanged as is the case in an ordinary Hilbert space. Of particular importance to us in this respect is the relation $\int_T x(t) d(P_t f, g) = (\{\int_T \overline{x(t)} dP_t\} f, g)$ established in Lemma 1 below. This paper is a continuation of [6] and [7].

2. Let $(\mathfrak{A}, |\ |)$ be a proper H^*-algebra [1] and let $\tau(\mathfrak{A}) = \{xy \mid x, y \in \mathfrak{A}\}$ be its trace class [8]. It was shown in [8] that $\tau(\mathfrak{A})$ is a Banach algebra with respect to a norm $\tau(\ )$ which is related to the norm $|\ |$ of $\mathfrak{A}$ by the identity $\tau(a^*a) = |a|^2$, $a \in \mathfrak{A}$. There is a trace tr defined on $\tau(\mathfrak{A})$ such that $\text{tr}\,(ab) = \text{tr}\,(ba) = (a, b^*)$ for all $a, b \in \mathfrak{A}$ (here (,) denotes the scalar product of $\mathfrak{A}$). We have $|a| \leq \tau(a)$ for each $a \in \tau(\mathfrak{A})$ and $\tau(ax) \leq \tau(a) \cdot |x|$, $\tau(xa) \leq |x|\, \tau(a)$ if $x \in \mathfrak{A}$.

If $a = b^*b$ for some $b \in \mathfrak{A}$ we shall say that a is positive and write "$a \geq 0$". In this case we have $\tau(a) = \text{tr}\, a$. Also one can show that the sum of two positive members of $\tau(\mathfrak{A})$ is positive. It follows from the fact $a \in \tau(\mathfrak{A})$ is positive if and only if the operator La: $x \to ax$ is positive in the sense that $(ax, x) \geq 0$ for all $x \in \mathfrak{A}$. Also it is true that $\text{tr}\,(ab) \geq 0$ if both $a \geq 0$ and $b \geq 0$ (in this case $a = x^*x$, $b = y^*y$ for some $x, y \in \mathfrak{A}$; thus: $\text{tr}\,(ab) = \text{tr}\,(x^*xy^*y) = \text{tr}\,(yx^*xy^*) \geq 0$).

Received March 25, 1969. This work was supported by N.S.F. Grant GP-7620.