ON $\mathcal{L}_2$-SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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1. Consider the second order self-adjoint linear differential equation:

$$(p(t)x')' - q(t)x = 0, \qquad t \geq 0, \tag{1}$$

where $p(t)$ is absolutely continuous and positive, and $q(t)$ is locally integrable. We are here concerned with the existence of a non $\mathcal{L}_2$ solution to equation (1), i.e. whenever equation (1) is not of limit cycle type. When $p(t) \equiv 1$, two well-known criteria due respectively to Weyl [12] and Hartman [6] state that if (i) $q(t) > 0$, or (ii) $q \in \mathcal{L}_2[0, \infty)$, then equation (1) is not of limit cycle type. In fact, their results remain valid for arbitrary $p(t)$. The purpose of this note is to extend the two results mentioned above to the more general n-th order equation:

$$p_n(p_{n-1} \cdots \{p_1[p_0x]'\}' \cdots)' - q(t)x = 0, \quad t \geq 0, \tag{2}$$

where $p_0, p_1, \cdots, p_n$ are sufficiently smooth so that equation (2) admits a solution for every choice of initial values. Analogously, we say equation (2) is not of limit cycle type if not all solutions belong to $\mathcal{L}^2[0, \infty)$. We assume in addition that all p_i's are positive and p_0 is non-increasing. Our proposed extensions are the following two theorems:

Theorem 1. *If $q(t) > 0$, then equation* (2) *is not of limit cycle type.*

Theorem 2. *Let $p_{n-i} = p_i$, $i = 0, 1, 2, \cdots, n$. If $q(t) \in \mathcal{L}^2[0, \infty)$, then equation* (2) *is not of limit cycle type.*

For convenience, we introduce the differential operators D_i, $i = 0, 1, 2, \cdots, n$, defined inductively by $D_0x = p_0x$, $D_ix = p_i(D_{i-1}x)'$, $i = 1, 2, \cdots, n$. In this notation, equation (2) takes the simple form $D_nx = qx$.

Proof of Theorem 1. Consider the solution $x(t)$ of (2) defined by the initial conditions $D_ix(0) = 1$, $i = 0, 1, 2, \cdots, n - 1$. Since $D_0x(0) = 1$ and $(D_0x)'(0) > 0$, hence $D_0x(t) > 1$ in some right neighborhood of $t = 0$. We first prove that $D_0x(t) > 1$ for all $t > 0$. Assume the contrary, then there must exist $T > 0$ such that $D_0x(t) > 1$ for all $t \in (0, T)$ and $D_0x(T) = 1$. Denote the compact interval $[0, T]$ by I, $\eta = \inf_{t \in I} q(t)$, and $\rho_k = \sup_{t \in I} p_k(t)$. From equation (2), we obtain $(D_{n-1}x(t))' \geq \eta/\rho_0^2$ for all $t \in I$. Using the definition of D_i's, we obtain inductively $(D_ix(t))' \geq 1/\rho_{i+1}$ $i = n - 2, \cdots, 2, 1$; and finally $(D_0x(t))' \geq 1/\rho_1 > 0$, from which we conclude that $D_0x(T) > 1$, which is a contradiction. From the fact that $p_0(t)$ is non-increasing, and $D_0x(t) \geq 1$

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