SPLITTING TOPOLOGICAL SEMIGROUPS

By T. H. McH. Hanson

Introduction. In [5] we find a discussion of splitting in topological groups. This raises the question as to whether or not there is something similar for topological semigroups. In this paper we give a definition of splitting in topological semigroups with identity which is shown to be a generalization of the one for topological groups. Using this, we prove an analogue of the Iwasawa Theorem for a special class of semigroups; this is Theorem I. The last results of this paper use Theorem I to characterize some semigroups.

All topological spaces are to be Hausdorff. If A is a subset of the space X, then A^* will denote the closure of A.

The author gratefully acknowledges the very helpful comments of the referee which resulted in both sharper results and shorter proofs.

Preliminaries. If S is a topological semigroup with identity, and T is a closed, normal subsemigroup of S which contains 1, the identity, T is said to *split* S if there is a closed subsemigroup K of S, with 1 in K, such that the map $(t, k) \mapsto tk$ is a homeomorphism from $T \times K$ onto S. In this situation, we say S is the *semi-direct product* of T with K.

If G is a group, and N is a subgroup of G such that $G = NK$ for some semigroup K contained in G with 1 in K, and $N \cap K = \{1\}$, then K is also a group. From this one sees that splitting in topological semigroups with identity, as given above, corresponds to the definition for groups, whenever S is in fact a group and T is a closed normal subgroup. Our first result is nothing too surprising. Its proof is straight-forward, and is therefore omitted.

Proposition 1. *Suppose S is a topological semigroup with identity, T is a subsemigroup of S which contains* 1, *and S is the semi-direct product of T with K. If K is normal, the product is direct.*

Let P denote the multiplicative group of positive real numbers, and P^- the multiplicative semigroup of non-negative reals. From [1], we set $V_n^- = P^- \times P^- \times \cdots \times P^-$ (n-copies), and we set $V_n = P \times P \times \cdots \times P$ (n-copies); V_n is a Lie group, and V_n^- is a topological semigroup. We let e denote the zero of V_n^-. If V_n^- acts as a transformation semigroup on the topological space X, we set $X' = \{x \mid s \mapsto sx\colon V_n^- \to V_n^-x \text{ is } 1 - 1\}$. In [1] it is shown that if X is locally compact, X' is open in X. Also, X' is invariant under the action of V_n on X induced by the action of V_n^-.

Received February 3, 1969; in revised form August 25, 1969. This work is part of the author's doctoral dissertation written under the direction of Professor J. G. Horne at The University of Georgia.