SINGULAR MAPS OF MANIFOLDS

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1. Introduction. A map $f: M \to N$ is *proper* if for each compact set $K \subset N$, $f^{-1}(K)$ is compact. It is *topologically equivalent* to $g: X \to Y$ if there exist homeomorphisms $\alpha: M \to X$ and $\beta: N \to Y$ such that $\beta f \alpha^{-1} = g$.

1.1. Definition. Given a map $f: M \to N$ and $x \in M$, let F be the component of $f^{-1}(f(x))$ containing x. The *singular set* A_f is defined as follows: $x \in M - A_f$ if and only if there exist neighborhoods U of F and V of $f(x)$ such that $f \mid U$: $U \to V$ is topologically equivalent to the product projection map of $V \times F$ onto V.

Let M^n and N^p be manifolds of dimensions n and p respectively, and without boundary unless otherwise indicated. If a map f: $M^n \to N^p$ is said to be differentiable of order m, then it will be understood that M^n and N^p are differentiable manifolds also of order m.

Given maps ψ: $M \to S$, ω: $E \to X$, define $\psi \times \omega$: $M \times E \to S \times X$ by $\psi \times \omega(x, t) = (\psi(x), \psi(t))$. Define the *open cone* $c(M)$ as the identification space obtained from $M \times [0, 1)$ by identifying $M \times \{0\}$ to a point; let ι be the identity map on $[0, 1)$, and let the *cone map* $c(\psi)$: $c(M) \to c(S)$ be the map induced by $\psi \times \iota$. If $M = \emptyset$ we will consider $c(M)$ to be a single point.

1.2. Theorem. *Let $f: M^n \to N^p$, $n \geq p$, be a proper C^p map and $\bar{H}^{n-p}(f^{-1}(y) \cap A_f; Z_2) = 0$ for each $y \in N^p$. Then there exists a closed set $Y \subset f(A_f)$ with* dim $Y <$ max (0, dim $f(A_f)$) *so that if $y \in f(A_f) - Y$ and $F \subset A_f$ is a component of $f^{-1}(y)$, then for each open neighborhood W of F there exist neighborhoods $U \subset W$ of F and V of y so that $f \mid U$ is topologically equivalent to $(c(\psi) \times \iota_{p-m})\lambda$, where*

(i) *$\lambda: U \to c(K^{n-p+m-1}) \times E^{p-m}$ is a monotone map with $A_\lambda \subset A_f$ and $K^{n-p+m-1}$ a manifold (or empty);*

(ii) *$c(\psi) \times \iota_{p-m}: c(K^{n-p+m-1}) \times E^{p-m} \to c(S^{m-1}) \times E^{p-m}$, where $\psi: K^{n-p+m-1} \to S^{m-1}$ is a bundle map (with possibly empty fiber) and ι_{p-m} is the identity map on E^{p-m}.*

The proof of (1.2) is by induction and will appear as a series of lemmas in §3, the last of which is (3.17).

The basic techniques and notation used here are as in [3], but in that paper $f^{-1}(y) \cap A_f$ is assumed to be zero-dimensional and stronger conclusions are proved about the maps λ and ψ. If $n = p$, then the cohomology condition implies that $A_f = \emptyset$ so the conclusions are satisfied vacuously; P. T. Church has proved [2] a very strong factorization theorem for this case with no cohomology condition.

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