ALGEBRAS OF MEASURABLE FUNCTIONS

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This paper concerns abstract descriptions of the algebras of real-valued functions on a set measurable with respect to a σ-algebra of subsets, and of the algebras obtained by reducing modulo an ideal of null functions. While there is a reasonably large literature with approximately this point, most of the results heavily involve the fact that these structures are σ-complete as lattices. However, B. Brainerd has obtained some results which, when coupled with some theorems of him and F. W. Anderson, imply purely algebraic descriptions of these structures; the proofs which are obtained from the relevant papers are rather involved, and use σ-completeness. What we shall do here, principally, is give reasonably short and direct proofs of these theorems, the proofs not using σ-completeness at all, and then briefly derive from these the descriptions involving σ-completeness. (A precise statement of the results of Anderson and Brainerd, and a brief survey of some other literature, appear in §4 below.)

1. Background. The characterizations in question will be carried out in the context of a class of lattice-ordered algebras studied by M. Henriksen and D. G. Johnson in [7]; these are called "uniformly closed ϕ-algebras," and for them the lattice structure is completely determined by the algebra operations [7, 3.8]. In an effort to make the present paper reasonably self-contained, we recall some relevant definitions and results.

All topological spaces will be completely regular Hausdorff, and most will be compact. With X a space, let $D(X)$ be the set of continuous functions f on X to the two-point compactification of the reals, $R \cup \{\pm\infty\}$, which are real-valued on a dense subset $\mathcal{R}(f)$ of X. With $f, g \in D(X)$ and $r \in R$, $f \vee g$, $f \wedge g$, and rf are defined pointwise, and are members of $D(X)$. With f, g, $h \in D(X)$, we write $h = f + g$ if $h(x) = f(x) + g(x)$ for all $x \in \mathcal{R}(f) \cap \mathcal{R}(g)$; $h = f \cdot g$ is defined similarly. In general, sums and products of members of $D(X)$ need not exist.

A ϕ-algebra is a lattice-ordered algebra over R, which is archimedean, and which has an identity 1 which satisfies: $a \wedge 1 = 0$ implies $a = 0$. It is not hard to see that a subset of a $D(X)$ which is a lattice and algebra under the operations discussed above, and which contains the constant function 1, is a ϕ-algebra.

1.1 [7, 2.3]. A ϕ-algebra A is isomorphic (as an algebra and lattice) to a ϕ-algebra of functions in $D(\mathfrak{M}(A))$, where $\mathfrak{M}(A)$ is the compact space of maximal absolutely convex ring ideals of A carrying the Stone topology. The isomor-

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