CONTRACTED IDEALS IN KRULL DOMAINS

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In [3], Gilmer and Mott prove the following result. (See Remark 6 of [3]; also, consult [3] for the definitions of properties C and ξ.)

Suppose that D is a Prüfer domain, that S is a unitary overring of D in which each nonzero element of D is regular, and that D has property C with respect to S. Then D has property ξ with respect to S.

The purpose of this paper is to prove that this result from [3] remains valid if the condition that D is a Prüfer domain is replaced by the assumption that D is a Krull domain—that is, an integral domain which can be written as the intersection of a family $\{V_\lambda\}_{\lambda \in \Lambda}$ of rank one discrete valuation overrings of D such that each nonzero element of D is a nonunit in only finitely many V_λ's. The basic theory of Krull domains is given in [2; §35] and in [6; §33], and we shall use freely the results on Krull domains contained in these two references.

Our first lemma uses this terminology: If A and B are ideals of a ring R, we say that B *is prime to* A if $A : B = A$; if $b \in R$, then b *is prime to* A is defined to mean $A : b = A$, while b *is prime to* a, where $a \in R$, means $(a) : b = (a)$. If A admits a shortest representation $A = \bigcap_{i=1}^{n} Q_i$ in R, where Q_i is P_i-primary, and if B is finitely generated, then B is not prime to A if and only if $B \subseteq P_i$ for some i. (Compare with [8; 36].) If R has an identity and if a and b are regular in R, then b prime to a implies that a is prime to b, and either condition is equivalent to the validity of the equality $(a) \cap (b) = (ab)$. In particular, if D is a Krull domain and if $\{P_\lambda\}_{\lambda \in \Lambda}$ is the set of minimal prime ideals of D, then given $d \in D - \{0\}$, (d) has a unique shortest representation as an intersection of symbolic powers of a finite set of P_λ's; thus if $a, b \in D - \{0\}$, then a is prime to b if and only if a and b belong to no common P_λ.

Lemma 1. *Suppose that R is a ring with identity and that $f = a_1X_1 + \cdots + a_nX_n - a$ and $g = b_1X_1 + \cdots + b_nX_n - b$ are elements of $R[X_1, \cdots, X_n]$ such that a_1 and b_1 are regular in R and a_1 is prime to b_1. Then any solution $X_i = r_i$, $2 \le i \le n$, of the equation $b_1f - a_1g = 0$ over R determines a unique value r_1 of X_1 such that $X_i = r_i$, $1 \le i \le n$, is a solution of the system $f = g = 0$ over R.*

Proof. By hypothesis, $t = b_1(a - a_2r_2 - \cdots - a_nr_n) = a_1(b - b_2r_2 - \cdots - b_nr_n) \in (b_1) \cap (a_1) = (b_1a_1)$—say $t = r_1b_1a_1$, where $r_1 \in R$. Then since a_1 and b_1 are regular in R, $a - a_2r_2 - \cdots - a_nr_n = r_1a_1$ and $b - b_2r_2 - \cdots - b_nr_n = r_1b_1$ so that $X_i = r_i$, $1 \le i \le n$, is a solution of the system $f = g = 0$

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