A SIMPLIFIED PROOF OF CHEBYSHEV'S THEOREM

By Nina Spears

1. Introduction. For a real variable $x \geqq 1$ and a positive integer n let

$$\pi(x) = \sum_{p \le x} 1, \qquad \text{the number of primes less than or equal to } x,$$

and

$$\Psi(x) = \sum_{n \le x} \Lambda(n),$$

with

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, p \text{ prime}, \quad m \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

Chebyshev's Theorem asserts that there exist positive constants a and A such that for $x \geq 2$,

$$a < \frac{\pi(x)}{x/\log x} < A; \tag{1}$$

that is, for sufficiently large values of x, $(\pi(x) \log x)/x$ is bounded above, and is bounded away from 0.

Due to a well known result of Chebyshev, the above is equivalent to asserting the existence of positive constants b and B such that for $x \geq 2$,

$$b < \frac{\Psi(x)}{x} < B. \tag{2}$$

And in fact, Shapiro [6] has shown that the left inequality in (2) is deducible from the right inequality in (2), making Chebyshev's Theorem derivable from

$$\Psi(x) = O(x) \qquad (x \geq 1). \tag{3}$$

Several proofs of (3) are available, those mentioned here incorporating the techniques from convolution theory, a by-product of the Selberg proof of the prime number theorem [5]. The first application of the theory of convolutions was made by Yamamoto [8] in 1955, indicating the recentness of its contribution to the Chebyshev theory. For a proof of (3) prefaced with a development of the symbolic apparatus of convolution theory, the reader is referred to Ayoub's work [1; 133–134]. The presentation by Gelfond and Linnik [2] is a "long hand" version of Ayoub's proof, not making use of the algebra of convolutions.

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