RELATIVE ANNIHILATORS IN LATTICES

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1. Introduction. As a natural generalization of the pseudo-complement a^*b of an element a of a lattice relative to an element b, we introduce the *annihilator* $\langle a, b\rangle$ *of a relative to b*; it is the family of all elements x such that $a \cap x \leq b$. The greatest element of $\langle a, b\rangle$, if it exists, is the relative pseudo-complement a^*b (see [1; 147]). Thus a lattice is relatively pseudo-complemented if and only if each annihilator has a greatest element, and hence is a principal ideal. It will be shown that each annihilator is an ideal if and only if the lattice is distributive, and we also give a weaker condition on annihilators that is equivalent to modularity.

The main results concern distributive lattices L in which $\langle a, b\rangle \cup \langle b, a\rangle = L$, identically, i.e., for any elements a and b of L, the join of the annihilator ideals $\langle a, b\rangle$ and $\langle b, a\rangle$ in the lattice of ideals of L is always the improper ideal L itself. This condition is analogous to the condition $a^* \cup a^{**} = 1$ for a Stone lattice, where a^* denotes the pseudo-complement of a; see [1; 149, Problem 70], [2], [3], [7], [9], [10] and [15], [16].

It will be shown that the annihilator condition $\langle a, b\rangle \cup \langle b, a\rangle = L$ in a distributive lattice L is satisfied if and only if the filters of L containing any given prime filter form a chain.

Examples are given of distributive lattices with 0 and 1 in which $\langle a, b\rangle \cup \langle b, a\rangle = L$ but which are not pseudo-complemented; they are provided by the same lattices which motivated this paper, the lattices of zero-sets of real-valued continuous functions on a topological space. Examples are also given among lattices of closed sets and lattices of cozero-sets.

2. Characterizations.

Theorem 1. *For any lattice L, the following are equivalent.*

(1) *L is distributive.*

(2) *$\langle a, b\rangle$ is an ideal for all a and b.*

(3) *$\langle a, b\rangle$ is an ideal whenever $b \leq a$.*

Proof. Let L be distributive. If $x \in \langle a, b\rangle$ and $y \leq x$, then $a \cap y \leq a \cap x \leq b$, and hence $y \in \langle a, b\rangle$. (Note that this part does not require distributivity.) If $x, y \in \langle a, b\rangle$, then $a \cap x \leq b$ and $a \cap y \leq b$; hence $a \cap (x \cup y) = (a \cap x) \cup (a \cap y) \leq b$, and thus $x \cup y \in \langle a, b\rangle$. Hence $\langle a, b\rangle$ is an ideal.

Now let $\langle a, b\rangle$ be an ideal whenever $b \leq a$. Let $x, y, z \in L$. Clearly $(x \cap y) \cup (x \cap z) \leq x$, and hence the annihilator $\langle x, (x \cap y) \cup (x \cap z)\rangle$ is an ideal. Since $x \cap y \leq (x \cap y) \cup (x \cap z)$, it follows that y (and similarly z) belongs to

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