THE SUSLIN-KLEENE THEOREM FOR COUNTABLE STRUCTURES

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Perhaps the most important single result of hierarchy theory is the theorem of Kleene which identifies the classes of hyperarithmetical and Δ_1^1 sets of integers. (This was announced in [3] and proved in [4]. It is often called the Suslin–Kleene theorem since it is the constructive analog of Suslin's theorem that identifies the Borel with the analytic-coanalytic sets of reals.) A set is Δ_1^1 if it can be defined by formulas $\exists \alpha P(\alpha, n)$ and $\forall \alpha Q(\alpha, n)$, where α ranges over number-theoretic functions and $P(\alpha, n)$, $Q(\alpha, n)$ are arithmetical. If the theorem is to be non-trivial, then "hyperarithmetical" must be given a "constructive" definition, e.g. Kleene's original definition in [3], or "recursive in **E**" in the sense of [5], or "inductively definable with inductively definable complement" in the sense of [9]. In this form the theorem has been called a *construction principle* (see [1]), since it allows us to "construct" the Δ_1^1 sets by iterating essentially first-order operations.

The theory of hyperarithmetical sets has been generalized to arbitrary first-order structures in the sequence of papers [6]–[8], but the generalization of this fundamental result fails. In the joint paper [2] the result is proved for structures of the form $\langle A, \epsilon \rangle$, where A is a *countable* transitive set closed under pairing, and the proof uses the completeness theorem for certain infinitary logics. Here we give a proof for all *countable* structures satisfying some mild definability conditions. Our proof is very similar to Kleene's original proof, except for one new trick.

If we generalize the approach to hyperarithmetical sets through inductive definitions, then our proof is completely elementary and does not use any of the abstract theory of hyperprojective sets; thus this paper can be understood by anyone who knows the language of lower predicate calculus and the rudiments of model theory—except for §8 where we relate this result to the theory in [6]–[8]. (I must thank C. C. Chang who challenged me to prove the theorem to him in an hour if it were as easy as I claimed and thus made me realize the elementary character of the proof.) The equivalence of this approach to hyperarithmetical sets with suitable generalizations of all other known "constructive" approaches is shown in [6]–[8]; see also §8 of this paper.

The precise definitions are given in §1–§3, the main result is stated in §3 and the proof is given in §4–§5. In §7 we show that the classical representation of π_1^1 sets in terms of the property of well-foundedness does not hold in the generality in which we prove the main result, so that use of some new trick is essential.

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