NON-ARCHIMEDEAN BANACH ALGEBRAS

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In this paper the algebraic properties of the rings $V = \{x \in X \mid ||x|| \leq 1\}$ and $W = \{x \in X \mid \inf_{m \in M} ||x + m|| \leq 1$ for all maximal ideals M of $X\}$, where X is a commutative non-Archimedean Banach algebra with identity, are investigated. In the first section, the structure of V and W is related to the structure of X; in the second section, the maximal ideals of V are considered; and, in the third section, the relations between V and W are considered, and, in particular, a condition on the set of idempotent elements of a semi-simple algebra X is given which implies the useful equality $V = W$.

The following notation will be used throughout this paper:

1. X is a commutative non-Archimedean Banach algebra with identity e over the complete non-trivially valued field F; i.e.,

(a) X is a normed linear space satisfying $||x + y|| \leq \max(||x||, ||y||)$,
(b) X is complete,
(c) $||xy|| \leq ||x|| \, ||y||$,
(d) $||e|| = 1$

(a space satisfying (a), (c) and (d) will be referred to as a normed algebra). The isometric isomorphism $\alpha \leftrightarrow \alpha e$ will be used, when convenient, to identify F as a subset of X.

$\mathfrak{M}$ is the set of all maximal ideals of X; $x(M)$, where $M \in \mathfrak{M}$ and $x \in X$, is the coset of M which x belongs; and $||x(M)|| = \inf_{m \in M} ||x + m||$ is used to make X/M a Banach algebra. X/M is always assumed to be an extension field of F under the identification $\alpha \leftrightarrow \alpha e(M)$. $\mathfrak{M}' = \{M \in \mathfrak{M} \mid X/M = F\}$. $\mathfrak{R}(X)$ is the Jacobson radical, $\bigcap \mathfrak{M}$.

2. $\mathfrak{o} = \{\alpha \in F \mid |\alpha| \leq 1\}$, the ring of integers of F; $\mathfrak{p} = \{\alpha \in F \mid |\alpha| < 1\}$; and for $s > 0$, $\mathfrak{p}_s = \{\alpha \in F \mid |\alpha| < s\}$ and $\mathfrak{p}_{s+} = \{\alpha \in F \mid |\alpha| \leq s\}$. For convenience we write $s < s+$, and let the index r stand for s or $s+$.

3. $V = \{x \in X \mid ||x|| \leq 1\}$, the *ring of integers of* X, and $P = \{x \in X \mid ||x|| < 1\}$; $W = \{x \in X \mid ||x(M)|| \leq 1$ for all $M \in \mathfrak{M}\}$ and $H = \{x \in X \mid ||x(M)|| < 1$ for all $M \in \mathfrak{M}\}$; $W_M = \{x \in X \mid ||x(M)|| \leq 1\}$ and $H_M = \{x \in X \mid ||x(M)|| < 1\}$. P_r, H_r and $H_{M,r}$ are defined in the same fashion as $\mathfrak{p}_r$. Certainly $V \subset W$ and $H_r = \bigcap_{M \in \mathfrak{M}} H_{M,r}$.

4. V_M will denote the ring of integers of X/M, and P_M will denote the set of elements of X/M with norm less than one; $V(M)$ and $P(M)$ are the images of V and P, respectively, under $x \to x(M)$. Then $V(M) \subset V_M$ and $P(M) = P_M$.

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