CONCERNING WHITNEY'S REPRESENTATIONS OF DIFFERENTIABLE EVEN AND ODD FUNCTIONS

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In a recent paper [2], I proved the following result concerning the differentiability of $F(x) = N(x)/D(x)$ where N and D have multiple order zeros at ξ.

THEOREM 0. *Let* $0 \leq k \leq n$. *Suppose that the derivatives* $N^{(n)}$ *and* $D^{(n)}$ *are continuous on a neighborhood of* ξ, *that* $N^{(n+1)}(\xi)$ *and* $D^{(n+1)}(\xi)$ *exist, and that*

$$N(\xi) = N'(\xi) = \cdots = N^{(k)}(\xi) = 0$$
$$D(\xi) = D'(\xi) = \cdots = D^{(k)}(\xi) = 0 \neq D^{(k+1)}(\xi).$$

If $F(\xi) = N^{(k+1)}(\xi)/D^{(k+1)}(\xi)$ *and* $F(x) = N(x)/D(x)$ *for* $x \neq \xi$, *then* $F^{(n-k)}$ *exists and is continuous on some neighborhood of* ξ.

In this result F may be complex-valued, but I take x and ξ to be real. The proof rested on Lemma 2 below and this, in turn, was an easy consequence of the following result which I proved from Taylor's theorem with integral remainder.

LEMMA 1. *Let* $L^{(n)}$ *be continuous on a neighborhood of* ξ, *and let* $L^{(n+1)}(\xi)$ *exist. Then there exists a function* Ψ *such that:*

(a) $$L(x) = \sum_{j=0}^{n+1} \frac{L^{(j)}(\xi)}{j!}(x-\xi)^j + \Psi(x)$$

(b) $\Psi^{(n)}$ *exists and is continuous on some neighborhood of* ξ

(c) $\Psi^{(k)}(x) = o(x-\xi)^{n+1-k}$ *as* $x \to \xi$ *for* $k = 0, 1, \cdots, n$.

LEMMA 2. *Let* $L^{(n)}$ *be continuous on a neighborhood of* ξ, *and let* $L^{(n+1)}(\xi)$ *exist. Let* $0 \leq k \leq n$, *and define* $\lambda(\xi) = L^{(k+1)}(\xi)/(k+1)!$ *but for* $x \neq \xi$ *let*

$$\lambda(x) = \left\{L(x) - \sum_{j=0}^{k} \frac{L^{(j)}(\xi)}{j!}(x-\xi)^j\right\} \Big/ (x-\xi)^{k+1}.$$

Then $\lambda^{(n-k)}$ *exists and is continuous on some neighborhood of* ξ. *Moreover,*

(d) $$\lambda^{(m)}(\xi) = \frac{m!}{(m+k+1)!} L^{(m+k+1)}(\xi) \quad \textit{if} \quad 0 \leq m \leq n-k$$

(e) $$\lim_{x\to\xi} (x-\xi)^{m+k-n} \lambda^{(m)}(x) = 0 \quad \textit{if} \quad n-k+1 \leq m \leq n.$$

Here, the result (d) was stated in Remark 4 of the earlier paper; and (e) is a consequence of the result proved there that $\varphi^{(m)}(x) = o(x-\xi)^{n-k-m}$ where $\varphi(x)$ is $\lambda(x)$ plus a polynomial of degree not exceeding $n - k \leq m - 1$. Clearly $\lambda^{(n)}$ is continuous on a deleted neighborhood of ξ.

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