AN ELEMENTARY APPROACH TO DIOPHANTINE EQUATIONS OF THE SECOND DEGREE

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If x is a root of the Diophantine equation $ax^2 + bx + c$, then x', where x' is defined by the relation $ax' = -ax - b$, is also a root, since $ax^2 + bx + c = x(ax + b) + c = x(-ax') + c = x'(-ax) + c = x'(ax' + b) + c = ax'^2 + bx' + c$. This simple observation is the basis for a very elementary, and apparently little-explored, approach to the solution of Diophantine equations in the second degree with any number of unknowns. In this paper we shall be principally concerned with integral solutions, although our methods also have application to the determination of rational solutions. Strictly speaking, a *solution* to a Diophantine equation is a solution in rational numbers; however, unless otherwise indicated, we shall use the terms solution and integral solution synonymously.

In §1, we use our approach to solve the binomial equation $\binom{x+1}{y} = \binom{x}{y+1}$ and we express its solutions in terms of Fibonacci numbers. At the same time, we prove a result (Theorem 1) which applies directly to a large class of 2-variable Diophantine equations. In §2, we discuss the situation for three variables. The notion of a planar equation is introduced and the property of being planar is characterized (Theorem 2). We then (§3) analyze in detail the planar equation $x^2 + y^2 + z^2 - xy - xz - yz - x - 2y - 3z = 0$, showing that its solution set consists of two infinite "planes" or triangular lattices of integers. In §4 we give an algorithm (Theorem 3) for solving directly a much wider class of 2-variable equations than that covered by Theorem 1. In fact, Theorem 3 applies to any equation one could reasonably expect to be amenable to our approach. The advantage of Theorems 1 and 3 over the traditional approach is that when they apply (as they do to most examples treated in the literature), the solutions are found very much faster, since it is not necessary to first perform a transformation on the equation. In §5 we solve all 2-variable Diophantine equations (by solving $u^2 - Dv^2 = k$ for any choice of the integers D, k) and compare our method to the classical one of Lagrange. In this way we show that Theorem 1 applies indirectly to any 2-variable equation, but only after a series of three transformations, the first two quadratic and the last linear.

1. The equation $\binom{x+1}{y} = \binom{x}{y+1}$; a useful 2-variable theorem. It is well known (see §5) that the solution in integers of any equation $ax^2+bxy+cy^2+dx+$

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