HANKEL DETERMINANTS FORMED FROM SUCCESSIVE DERIVATIVES

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Introduction. For Legendre polynomials $P_n(x)$, P. Turán proved the following inequality

$$(1)\qquad \begin{vmatrix} P_{n-1}(x) & P_n(x) \\ P_n(x) & P_{n+1}(x) \end{vmatrix} \le 0 \qquad n \ge 1, \quad -1 \le x \le 1.$$

This inequality was generalized in many directions by Beckenbach, Seidel and Szász [1] and Forsythe [2]. In a paper [3] Karlin and Szegö extended Turán's inequality to higher order determinants and also to other classes of orthogonal polynomials. They investigated the distribution of the zeros of the polynomial defined by the determinant of Hankel type (sometimes also called persymmetric determinant),

$$(2)\qquad H(n, m; x) = \begin{vmatrix} p(x) & p'(x) & \cdots & p^{(m-1)}(x) \\ p'(x) & p''(x) & \cdots & p^{(m)}(x) \\ \vdots & & & \vdots \\ p^{(m-1)}(x) & p^{(m)}(x) & \cdots & p^{(2m-2)}(x) \end{vmatrix} \qquad (n \ge m - 1)$$

where $p(x) = L_n^{(\alpha)}(x)$ is the Laguerre polynomial of degree n and of order α, $\alpha > -1$ and $p^{(i)}$ denotes the i-th derivative of $p(x)$. One of their many results was, that $H(n, m; x)$ has no zeros on the negative real axis, i.e. $H(n, m; x)$ keeps constant sign outside of the interval of orthogonality. They also conjectured that when $p(x)$ is the ultraspherical polynomial, the Hankel determinant (2) as a function of x keeps constant sign for $x > 1$ and $x < -1$.

In this paper we shall consider Hankel type determinants of order three where p is a polynomial restricted only in that all of its zeros lie in an interval $[a, b]$. We shall prove that all the zeros of $H(n, 2; x)$ lie in a region of the complex plane which does not contain points from that part of the real axis where $x < a$ and $x > b$. Similar results were obtained in [4] for Wronskians where elements are orthogonal polynomials. The method which we shall use can be illustrated in the simple case $m = 1$. Here

$$H(n, 1; x) = p(x)p''(x) - [p'(x)]^2 = [p(x)]^2 \left[\frac{p'(x)}{p(x)}\right]'.$$

(Of course the fraction is not defined at the zeros $x = \xi$ of p, but here and also in the sequel we suppose the function to be defined by letting x to approach to

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