NON-TANGENTIAL LIMITS FOR A SOLUTION OF THE HEAT EQUATION IN A TWO-DIMENSIONAL LIP$_\alpha$ REGION

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1. Introduction. This paper deals with the simple heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$ in a region described by the inequalities $0 < t < T$, $\eta(t) < x < \infty$, where η is a uniformly Hölder continuous function on $[0, T]$ with exponent $\alpha > \frac{1}{2}$. The problem is that of describing the behavior of u near the curved boundary, specifically, to investigate the relation between a type of boundedness near the boundary with a type of limiting behavior at the boundary. The precise definitions are given in §4. It will suffice to mention here that the boundedness and limiting behavior at a boundary point $(\eta(t_0), t_0)$ are both described in terms of what happens in parabolic regions of the form $\Gamma(t_0) = \{(x, t): |t - t_0| < \theta(x - \eta(t_0))^2, x > \eta(t_0)\}$, where $\theta > 0$. The word "non-tangential" in the title is taken in this sense and thus refers to behavior of u inside any such region. The more accurate terms "parabolic limit" and "parabolically bounded" are used in the definition in §4.

The principal result of the paper, Theorem 2, states that the notions of parabolic boundedness and parabolic limit are essentially the same (always neglecting a set of zero measure). This result should be regarded as an extension of results of Privaloff [10] for harmonic functions in two variables, of Calderón [1], who extended Privaloff's results by giving a proof independent of conformal mapping and valid for harmonic functions of n variables, and of Hattemer [5], who has results relating to the initial-value problem for the heat equation. A more general result for harmonic functions has been recently proved by Hunt and Wheeden [6], in which the result of Calderón for a flat boundary is extended considerably, Lipschitz regions figuring strongly in their analysis. Also for a flat boundary, Tu [11] has a corresponding theorem for solutions of the heat equation in several variables in a "spatial" half space.

This paper can be outlined as follows. In §3 a standard existence theorem for solutions of the relevant boundary-value problem for the heat equation is derived. This is only briefly described, since essentially much better results are available: as shown by Petrovskiĭ [9], the Lip$_\alpha$ condition ($\alpha > \frac{1}{2}$) on η is much stronger than necessary for the existence of barriers for a balayage procedure for solving this problem. Also, the case of an Lip$_\alpha$ curve is discussed in Gevrey [4], but there does not seem to be in print an actual statement of the "non-tangential" estimate of our Lemma 6. Our strong hypothesis on η insures among other things that the parabolas $\Gamma(t_0)$ are near their vertices contained

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