ALGEBRAIC KERNELS OF PLANAR SETS

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Throughout this paper X will be a compact subset of the complex plane $\mathbf{C}$ and I_L will be the ring of integers of any imaginary quadratic field L. In [1] Fekete defined the algebraic kernel $J(X, I_L)$ of X with respect to I_L to be the union of all complete sets of conjugates over L which are integral over I_L and are contained in X. The purpose of this paper is the explicit determination of $J(X, I_L)$. This seems to be difficult in general. We determine it for certain subsets of the closed unit disk D. The determination of the algebraic kernel is crucial in the problem of uniform approximation by polynomials with integral coefficients [2, 5.8].

In the course of the determination of J for arcs of the unit circle we found it necessary to determine explicitly the Galois groups of the cyclotomic fields over their quadratic subfields.

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It is easy to see that, if $z_0 \varepsilon I_L$, then $J(z_0 + X, I_L) = z_0 + J(X, I_L)$ and if z_0 is a unit of I_L then $J(z_0X, I_L) = z_0J(X, I_L)$. Thus the following results have a wider range of applicability than first appears.

Throughout the paper $\mathbf{Q}$ will denote the field of rational numbers and d will be the unique square free rational integer such that $L = \mathbf{Q}(\sqrt{d})$. Since L is imaginary, d is negative.

1. A simple case of the problem can be handled by the observation that if $\{z_1, \cdots, z_n\}$ is a complete set of conjugates integral over I_L, then the norm $z_1z_2 \cdots z_n \varepsilon I_L$. Indeed, suppose X is a compact subset of the closed unit disk D. Then $J(X, I_L) = (X \cap \{0\}) \cup J(X \cap T, I_L)$, where T is the unit circle.

It is clear that $(X \cap \{0\}) \subset J(X, I_L)$, so it suffices to prove that any nonzero element of $J(X, I_L)$ is in $J(X \cap T, I_L)$. Let $\{z_1, \cdots, z_n\}$ be a complete set of conjugates integral over I_L with $z_1 \neq 0$ and $\{z_1, \cdots, z_n\} \subset X$. Then $z_1z_2 \cdots z_n$ is the constant term of the minimal polynomial ρ of the z_i's over L and so is an element of I_L. Each $|z_i| \leq 1$, and if $|z_i| < 1$ for some i, then $|z_1z_2 \cdots z_n| < 1$ which implies $z_1z_2 \cdots z_n = 0$ by the explicit representation of elements of I_L [8; 234]. But ρ is irreducible and so must be simply $\rho(z) = z$. But this contradicts the assumption $z_1 \neq 0$. Thus $|z_i| = 1$ for all i and $z_1 \varepsilon J(X \cap T, I_L)$.

2. This result reduces the problem to finding $J(X, I_L)$ for subsets of the unit circle. Before proceeding with this we need the following generalization of a result due to Kronecker [5].

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