THE MINKOWSKI AND TCHEBYCHEF INEQUALITIES AS FUNCTIONS OF THE INDEX SET

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1. Introduction. Let I and J be nonempty disjoint finite sets of distinct positive integers; $\{a_i\}_{i \epsilon I \cup J}$ and $\{b_i\}_{i \epsilon I \cup J}$ sequences of complex numbers; and p and q real numbers such that $p > 1$ and $p^{-1} + q^{-1} = 1$. Let the set function $H(I)$ be given by

$$H(I) = (\sum_{i \epsilon I} |a_i|^p)^{1/p}(\sum_{i \epsilon I} |b_i|^q)^{1/q} - |\sum_{i \epsilon I} a_i b_i|.$$

Hölder's inequality asserts that $H(I) \geq 0$. W. N. Everitt, [1], has shown that, in fact,

$$H(I \cup J) \geq H(I) + H(J) \geq 0.$$

Roughly speaking, this last inequality asserts that the "Hölder difference function," $H(I)$, is a superadditive function of the index set.

Similar results concerning the arithmetic mean-geometric mean inequality have been given in [2], [5], [6], [10]; while other generalizations for general means are to be found in [2], [8], [7].

It is the purpose of the present paper to consider the Minkowski inequality (see, e.g., Hardy, Littlewood, and Pólya [4; 31]) and the Tchebychef inequality (see, e.g., [4; 43]) in light of the above results.

2. Minkowski's inequality. This inequality states that, under the assumptions indicated above, with $p > 1$ or $p < 0$ (in which case it will be assumed that a_i, b_i, and $a_i + b_i$ are nonzero),

$$(1) \qquad (\sum_{i \epsilon I} |a_i + b_i|^p)^{1/p} \leq (\sum_{i \epsilon I} |a_i|^p)^{1/p} + (\sum_{i \epsilon I} |b_i|^p)^{1/p};$$

while, if $0 < p < 1$, then the sense of this inequality reverses. It is natural to define a set function

$$D_1(I) = (\sum_I |a|^p)^{1/p} + (\sum_I |b|^p)^{1/p} - (\sum_I |a + b|^p)^{1/p},$$

where, for example, $\sum_I |a|^p$ denotes $\sum_{i \epsilon I} |a_i|^p$. One is then led to suspect that $D_1(I)$ might be a superadditive set function. However, Everitt [1], working with the integral formulation of Minkowski's inequality, has given examples to show that one cannot, in general, expect the difference function $D_1(I)$ to be "monotone" in I; and hence, not superadditive.

By raising both sides of (1) to the p-th power, one is led to consider the following set function:

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