EXISTENCE-UNIQUENESS THEOREMS FOR NON-LINEAR DIRICHLET PROBLEMS

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In this paper we consider Dirichlet problems for the equation

$$\nabla^2 u = f(u), \tag{1}$$

where ∇^2 is the Laplacian and f is non-linear. Most discussions of such problems involve the condition that u be defined and continuously differentiable for all u. However, in many applied problems, the physical meaning of u limits its range. The purpose of this paper is to discuss such restricted problems.

Let G be a bounded domain in R^n with boundary Γ. We consider the Dirichlet problems for the non-linear equations

$$\nabla^2 u = \frac{c_1 u}{1 + c_2 u}, \quad u \geq 0, \quad \text{in } G, \tag{2}$$

and

$$\nabla^2 u = -\frac{c_1}{1 + c_2 u}, \quad u \geq 0, \quad \text{in } G, \tag{3}$$

with boundary condition

$$u = \phi \geq 0 \text{ on } \Gamma, \tag{4}$$

where c_1 and c_2 are positive constants and ϕ is continuous. Courant [1] has proved the following theorems.

Theorem 1. *If f is defined, continuously differentiable and bounded for all u, then the Dirichlet problem for Equation* (1) *with boundary condition* (4) *has a solution in G.*

Theorem 2. *The Dirichlet problem for the equation*

$$\nabla^2 u + cu = 0,$$

where $c \leq 0$ and continuous, has at most one solution.

With the aid of these two theorems and the Maximum Principle, we are able to establish the following results.

Theorem 3. *The Dirichlet problem for Equation* (2) *with boundary condition* (4) *has one and only one solution.*

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